LIFTING SURFACE THEORY FOR WINGS OF ARBITRARY PLANFORM

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THESIS

LIFTING SURFACE THEORY FOR
WINGS OF ARBITRARY PLANFORM

by

John Leroy Parks

March 1976

Thesis Advisor: T.H. Gawain

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by

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ABSTRACT

This theory permits the calculation of pressure distributions over a thin airfoil in steady, inviscid, incompressible flow; or given a desired chordwise pressure distribution the camber line of an ideal wing can be determined. The method treats the circulation about the wing as a continuous vortex sheet of variable strength covering the wing planform and trailing downstream to infinity. The results of the present numerical solution for the pressure distribution solution are not satisfactory. However the solution for the camber line of an ideal wing are in reasonable agreement with published two dimensional results.

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TABLE OF SYMBOLS

$\mathcal{R}$	aspect ratio
b	wing span
C	relative chord function
c	wing chord
$\bar{c}$	mean aerodynamic chord
c_l	section lift coefficient
C_L	wing lift coefficient
D	substantial derivative
F	influence functions
G	influence functions
h	auxiliary functions
(i,j,k)	Cartesian unit vectors
H	chordwise pressure functions
$\ddot{n}$	normal vector
P	control point
q_∞	free stream dynamic pressure
R	chordwise summing limit
S	wing area or spanwise summing limit
s	wing semi-span
T	total number of control points
V_U	velocity vector on wing upper surface
V_L	velocity vector on wing lower surface
V_∞	free stream velocity vector
w	downwash velocity
(x,y,z)	Cartesian coordinates

X_L	wing leading edge function
X_T	wing trailing edge function
α	angle of attack
Γ	circulation function
γ	vortex sheet strength
ξ	chordwise relative coordinate
θ	spanwise angular coordinate
η	spanwise relative coordinate
$\sum$	summing symbol
ϕ	chordwise angular coordinate
$[\ ]$	matrix symbol
$\{\ \}$	vector symbol
$[\ ]^{-1}$	inverse of indicated matrix
∇	gradient operator

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I. INTRODUCTION

This analysis is a continuation of two dimensional thin airfoil theory [Ref. 1] and lifting line theory [Ref. 2] into a three dimensional lifting surface theory, using the concept of a continuous vortex sheet over the wing which trails off to infinity in the downstream direction. The continuous vortex sheet differentiates this method from the vortex lattice kernel methods of Multhopp [Ref. 3], Falkner [Ref. 4], Weissinger [Ref. 5], and Watkins et. al. [Ref. 6] and double-lattice finite element methods of Lopez and Shen [Ref. 7] and Giesing et. al. [Ref. 8]. The present method is restricted to steady, inviscid, incompressible flow and thin airfoils which are symmetrical in semispan.

The circulation about the wing is treated as generated by a continuous vortex sheet of variable strength covering the wing planform and trailing downstream to infinity. The vortex sheet strength is set by the Kutta condition at the trailing edge and requiring no flow through the wing at specified control points.

The flow equations for the wing are resolved into downwash angles, with assumed series solutions for the circulation functions. From the known planform geometry and the required solution at the control points, no flow through the wing, the constants of the circulation functions series are determined. The equations are put into matrix form, the matrices being formed by numerical summation over the planform, and solved by matrix algebra. Once the constants of the circulation functions are known all of the airfoil characteristics can be calculated.

This analysis generates the spanwise and chordwise pressure distributions over the wing which are necessary in designing the wing structure. The solution is also done in

reverse, that is the pressure distribution over the wing is specified and the shape of the mean camber line of an ideal airfoil is determined.

This analysis uses the consepts of additional and reference lift which is of academic interest in understanding airfoil theory.

II. General Problem Development

A. THE WING VORTEX SYSTEM

The present analysis uses the theory of thin wings to describe the characteristics of lifting surfaces in steady, inviscid, incompressible flow. The most common planforms are those which involve straight leading and trailing edges, as illustrated in Fig. 1, but the present analysis applies also to wings having curved leading and trailing edges. We consider only wings that are symmetrical with respect to the x axis. We also restrict the present analysis to lift distributions which are symmetrical with respect to the x axis.

The wing and the trailing vortex system associated with it are treated as a continuous vortex sheet of variable strength $\vec{\gamma}$. The sheet strength $\vec{\gamma}$ is regarded as a vector in the x,y plane. The wing vortex sheet is treated as lying in or very close to the x,y plane and $\vec{\gamma}$ is everywhere tangent to the vortex sheet.

The sheet strength $\vec{\gamma}$ is related to the velocities V_U and V_L just above and just below the vortex sheet at a given point x,y and to the remote velocity V_∞ , as indicated in the vector diagram, Fig. 2. From the vector diagram Fig. 2.

$$\vec{V}_U = \vec{V}_\infty + \frac{1}{2}\vec{\gamma} \times \vec{k} \tag{1.1}$$

$$\vec{V}_L = \vec{V}_\infty - \frac{1}{2}\vec{\gamma} \times \vec{k} \tag{1.2}$$

It then follows that

$$\frac{1}{2}\left(\vec{V}_U + \vec{V}_L\right) = \vec{V}_\infty \tag{1.3}$$

$$\left(\vec{V}_U - \vec{V}_L\right) = \vec{\gamma} \times \vec{k} \tag{1.4}$$

or conversely

$$\vec{\gamma} = \vec{k} \times (\vec{V}_U - \vec{V}_L) \tag{1.5}$$

In Fig. 3, consider the circulation $d\Gamma$ around contour AOBB′O′A′A. Line segment AOB lies just above the vortex sheet, and segment B′O′A′ lies just below the vortex sheet.

$$\overrightarrow{AOB} = \overrightarrow{A'O'B'} = \vec{ds} \tag{1.6}$$

Then

$$d\Gamma = \oint_C \vec{V} \cdot \vec{dr} = \int_A^B \vec{V} \cdot \vec{dr} + \int_B^{B'} \vec{V} \cdot \vec{dr} + \int_{B'}^{A'} \vec{V} \cdot \vec{dr} + \int_{A'}^{A} \vec{V} \cdot \vec{dr}$$

$$d\Gamma = \vec{V}_U \cdot \vec{ds} + 0 - \vec{V}_L \cdot \vec{ds} + 0 = (\vec{V}_U - \vec{V}_L) \cdot \vec{ds}$$

$$d\Gamma = (\vec{\gamma} \times \vec{k}) \cdot ds \tag{1.7}$$

It is assumed, which will subsequently be confirmed, that the $d\Gamma$ in equation (1.7) is an exact differential. Therefore, there exists a function

$$\Gamma = \Gamma(x, y)$$

such that for any small changes in the coordinates

$$d\Gamma = \left(\frac{\partial \Gamma}{\partial x}\right) dx + \left(\frac{\partial \Gamma}{\partial y}\right) dy \tag{1.8}$$

Translating this into vector terms gives:

$$d\Gamma = \left(\vec{i}\,\frac{\partial \Gamma}{\partial x} + \vec{j}\,\frac{\partial \Gamma}{\partial y}\right) \cdot \left(\vec{i}\,dx + \vec{j}\,dy\right) = \nabla\Gamma \cdot \vec{ds} \tag{1.9}$$

Comparing equations (1.7) and (1.9) shows

$$\nabla\Gamma = \vec{\gamma} \times \vec{k} \tag{1.10}$$

or conversely that

$$\vec{\gamma} = \vec{k} \times \nabla\Gamma \tag{1.11}$$

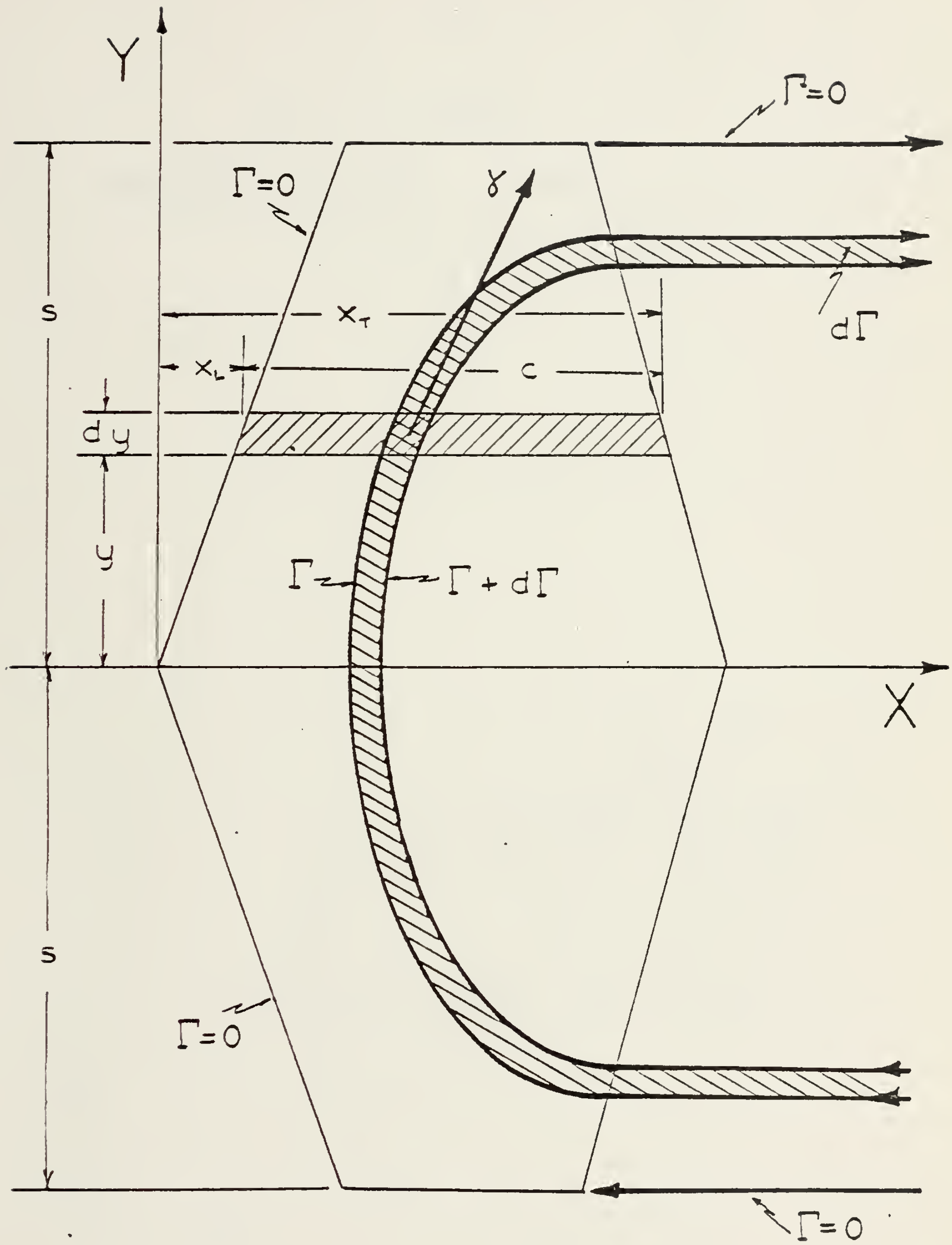

Fig. 1 Wing Coordinates

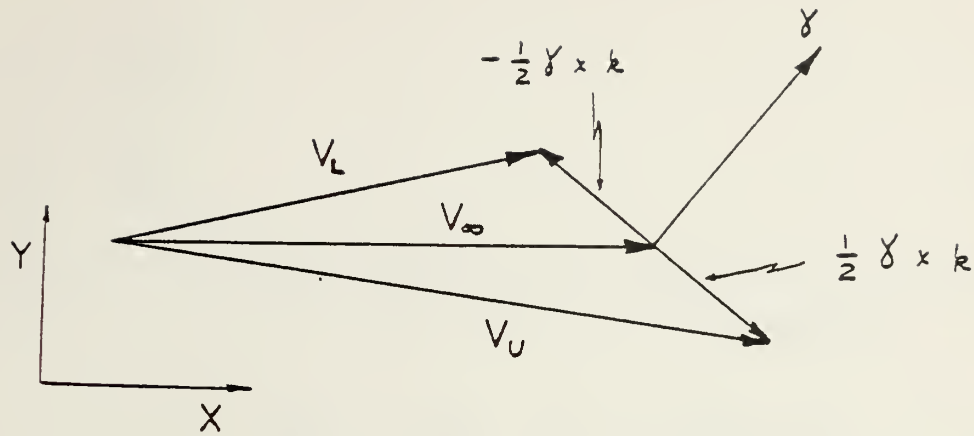

Fig. 2 Velocity Vectors

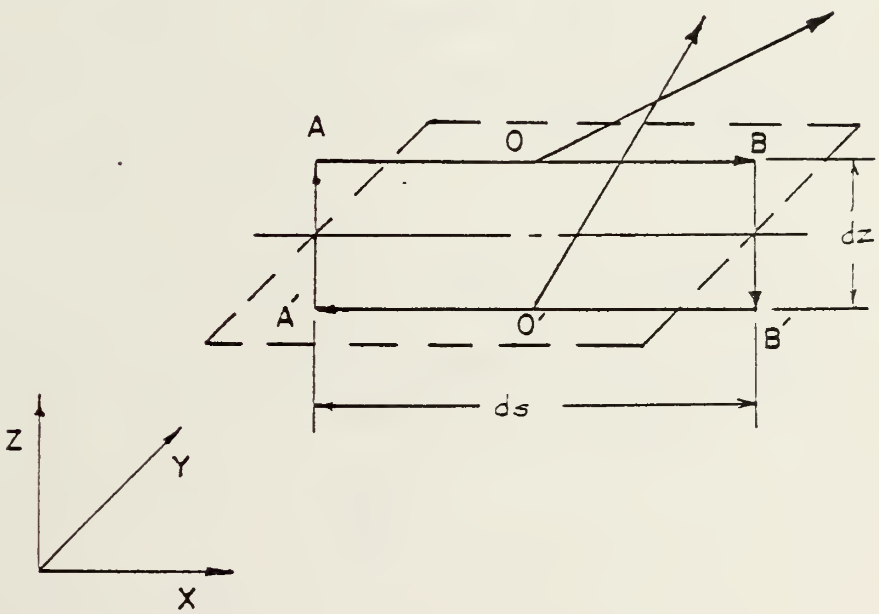

Fig. 3 Vortex Sheet Element

These results confirm that $d\Gamma$ is an exact differential.

Translating equation (1.11) into Cartesian components gives

$$\gamma_x = -\left(\frac{\partial \Gamma}{\partial y}\right) \tag{1.12}$$

$$\gamma_y = \left(\frac{\partial \Gamma}{\partial x}\right) \tag{1.13}$$

The vector function $\vec{\gamma}$ defined by equation (1.11) or by scalar equations (1.12) and (1.13) is non-divergent.

$$\nabla \cdot \vec{\gamma} = \frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \Gamma}{\partial y}\right) + \left(\frac{\partial \Gamma}{\partial x}\right) \equiv 0 \tag{1.14}$$

B. LIFT

Newton's Law applied to a fluid element of fixed mass $\rho \Delta x \Delta y \Delta z$ is

$$\vec{F} = \rho \Delta x \Delta y \Delta z \frac{D\vec{V}}{Dt} \tag{2.1}$$

It is assumed that the fluid is non viscous; that is, shearing stresses are absent. The resultant pressure force of the fluid element arising from the static pressure variation in the fluid, is the negative of the pressure gradient multiplied by the increment volume.

$$\text{resultant} = (\,-\,\text{grad } p\,)\,\Delta x \Delta y \Delta z$$

The pressure force plus the weight is the external force, therefore the equation of equilibrium becomes

$$\rho g - \text{grad } p = \rho \frac{D\vec{V}}{Dt} \tag{2.2}$$

which is Euler's equation.

If the fluid is assumed incompressible and equation (2.2) is integrated dropping the unsteady term it becomes Bernoulli's equation

$$\rho \frac{V^2}{2} + p - \rho g z = \text{constant} \tag{2.3}$$

The pressure difference across the wing at an arbitrary point can be found from Bernoulli's equation.

$$\left(\frac{P_L - P_U}{q_\infty}\right) = \left(\frac{\Delta P}{q_\infty}\right) = \left(\frac{V_U}{V_\infty}\right)^2 - \left(\frac{V_L}{V_\infty}\right)^2$$

$$= \frac{(\vec{V}_U + \vec{V}_L)\cdot(\vec{V}_U - \vec{V}_L)}{V_\infty^2} = \frac{2\vec{V}_\infty \cdot (\vec{\gamma} \times \vec{k})}{V_\infty^2}$$

$$= \frac{2\vec{i}\cdot(\vec{\gamma}\times\vec{k})}{V_\infty} = \frac{2\vec{i}\cdot\nabla\Gamma}{V_\infty} = \frac{2}{V_\infty}\left(\frac{\partial \Gamma}{\partial x}\right)$$

$$\left(\frac{\Delta P}{q_\infty}\right) = 4\left(\frac{\gamma_y}{2V_\infty}\right) \qquad (2.4)$$

Equation (2.4) ultimately fixes the lift distribution over the wing surface once the circulation function Γ has been found.

In the trailing vortex region behind the wing, there is no wing surface, and hence no pressure difference across the vortex sheet. According to the Kutta condition, the pressure difference drops to zero not only behind the wing but also all along the trailing edge. This is expressed as follows:

$$\left(\frac{\Delta P}{q_\infty}\right)_T = \frac{2}{V_\infty}\left(\frac{\partial \Gamma}{\partial x}\right)_T = 4\left(\frac{\gamma_y}{2V_\infty}\right)_T = 0 \qquad (2.5)$$

This result shows that the vortex lines behind the wing and along the trailing edge are all parallel to the x axis, with no spanwise components. Spanwise vorticity $\vec{j}\,\gamma_y$ can occur only along the wing surface.

In Fig. 4, next page, consider the incremental lift dL' exerted by the doubly shaded portion of the wing, that lies between y and (y+dy) and between x and x. Thus:

$$dL' = \frac{1}{2}\rho V_\infty^2 \int_{x_L}^{x} \left(\frac{\Delta P}{q_\infty}\right) dx\, dy$$

$$dL' = V_\infty \left[\int_0^{\Gamma} \frac{\partial \Gamma}{\partial x}\, dx\right] dy$$

$$dL' = \rho V_\infty \Gamma\, dy \qquad (2.6)$$

This result is a form of the Kutta-Joukowski Law which states

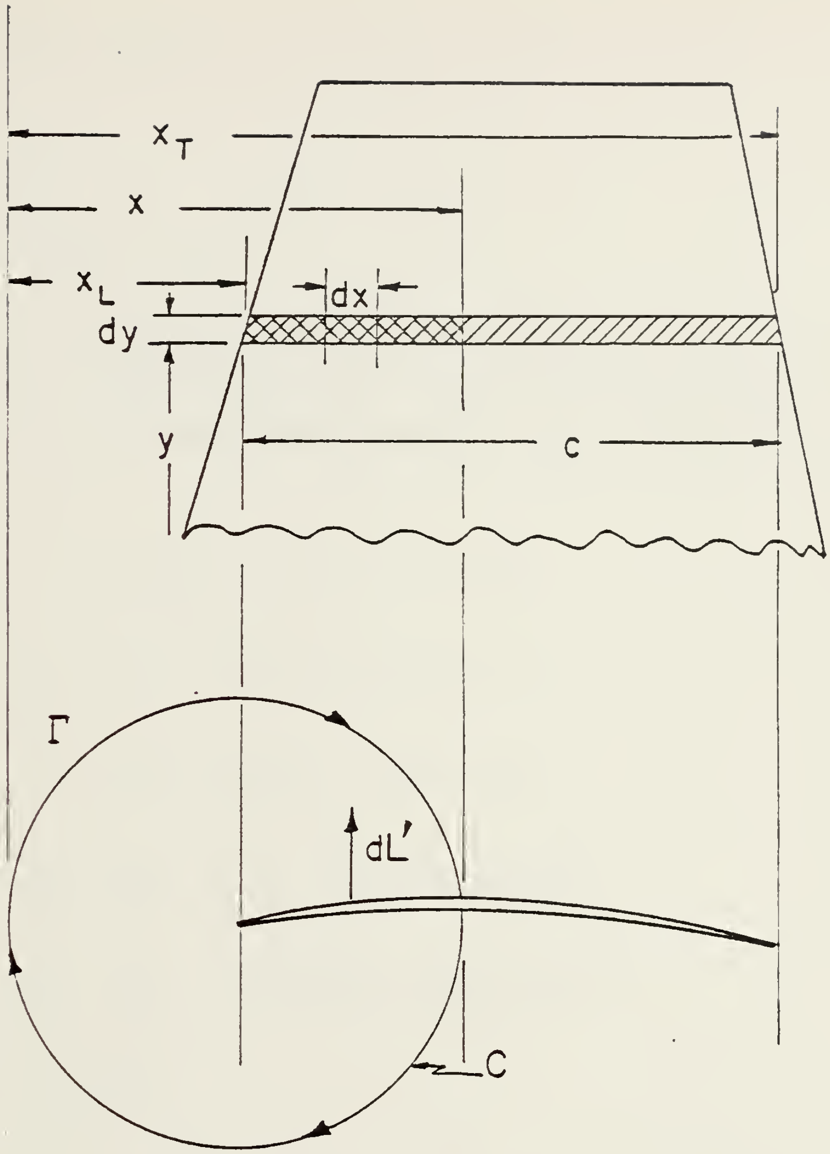

Fig. 4 Lift Integration

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that the force experienced by a body in a uniform stream is equal to the product of the fluid density, stream velocity, and circulation and has a direction perpendicular to the stream velocity. Here Γ represents the circulation about a contour C in the Plane y = constant. Contour C passes in front of the wing and pierces the wing surface at point x,y.

If we now extend the upper limit of integration to the trailing x_T, holding y constant, we obtain:

$$X \rightarrow X_T$$

$$\Gamma(x,y) \rightarrow \Gamma_T(y)$$

$$dL' = dL$$

$$dL = \rho V_\infty \Gamma_T \, dy \qquad (2.7)$$

thus Γ_T (y) fixes the spanwise lift distribution over the wing.

In order to reduce equation (2.7) to a convenient dimensionless form, it is useful to introduce the following auxiliary notation.

$$S = \text{total wing area, ft}^2 \qquad (2.8)$$

$$2s = b = \text{wing span, ft} \qquad (2.9)$$

$$\bar{c} = \frac{S}{b} = \text{mean chord, ft} \qquad (2.10)$$

$$AR = \frac{b}{\bar{c}} = \frac{b^2}{S} = \text{aspect ratio} \qquad (2.11)$$

$$\eta = \frac{y}{s} = \text{spanwise coordinate} \qquad (2.12)$$

$$d\eta = \frac{dy}{s} \qquad (2.13)$$

$$c_\ell = \frac{dL}{q_\infty c \, dy} = \text{section lift coefficient} \qquad (2.14)$$

$$C_L = \frac{L}{q_\infty S} = \text{wing lift coefficient} \qquad (2.15)$$

By utilizing the above nomemclature, it is easy to reduce the spanwise lift distribution as given in equation (2.7) to either of the following two equivalent forms.

$$\left(\frac{dL}{q_\infty S}\right) = dC_L = \frac{1}{2} c_\ell \left(\frac{c}{\bar{c}}\right) d\eta = \left(\frac{\Gamma_T}{V_\infty \bar{c}}\right) d\eta \qquad (2.16)$$

To find the total lift we integrate over the span, For symmetrical lift distribution it is only necessary to integrate over the semi-span and multiply by two. Thus:

$$C_L = \int_0^1 c_\ell \left(\frac{c}{\bar{c}}\right) d\eta = 2\int_0^1 \left(\frac{\Gamma_T}{V_\infty \bar{c}}\right) d\eta \qquad (2.17)$$

C. INDUCED VELOCITY

The velocity dw_P' induced at an arbitrary point P, coordinates x_p, y_p on the wing surface by an element of the vortex sheet which lies at the point whose coordinates are x,y is found as follows. The vortex sheet element has length ds, width dn and strength $\vec{\gamma}$. The circulation strength of the vortex filament involved is $d\Gamma = \gamma dn$.

The Biot Savart Law for this case may be written in the form:

$$d\vec{w}_P' = \frac{\vec{r} \times \vec{\gamma}\, dn\, ds}{4\pi r^3} \qquad (3.1)$$

where

$$\vec{r} = \vec{i}(x - x_p) + \vec{j}(y - y_p) + \vec{k}(z - z_p) \qquad (3.2)$$

dn ds = dS = element of the wing area

$$\vec{\gamma} = \vec{i}\,\gamma_x + \vec{j}\,\gamma_y + \vec{k}\,\gamma_z = \vec{k} \times \nabla\Gamma \qquad (3.3)$$

Neglecting the terms $\vec{k}(z-z_p)$ in equation (3.2) and $\vec{k}\,\gamma_z$ in equation (3.3).

$$\vec{\gamma} = -\vec{i}\left(\frac{\partial\Gamma}{\partial y}\right) + \vec{j}\left(\frac{\partial\Gamma}{\partial x}\right) \qquad (3.4)$$

also

$$\vec{r} \times \gamma = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x - x_p) & (y - y_p) & 0 \\ -\frac{\partial\Gamma}{\partial y} & \frac{\partial\Gamma}{\partial x} & 0 \end{vmatrix}$$

$$= \vec{k}\left[(x - x_p)\left(\frac{\partial\Gamma}{\partial x}\right) + (y - y_p)\left(\frac{\partial\Gamma}{\partial y}\right)\right] \qquad (3.5)$$

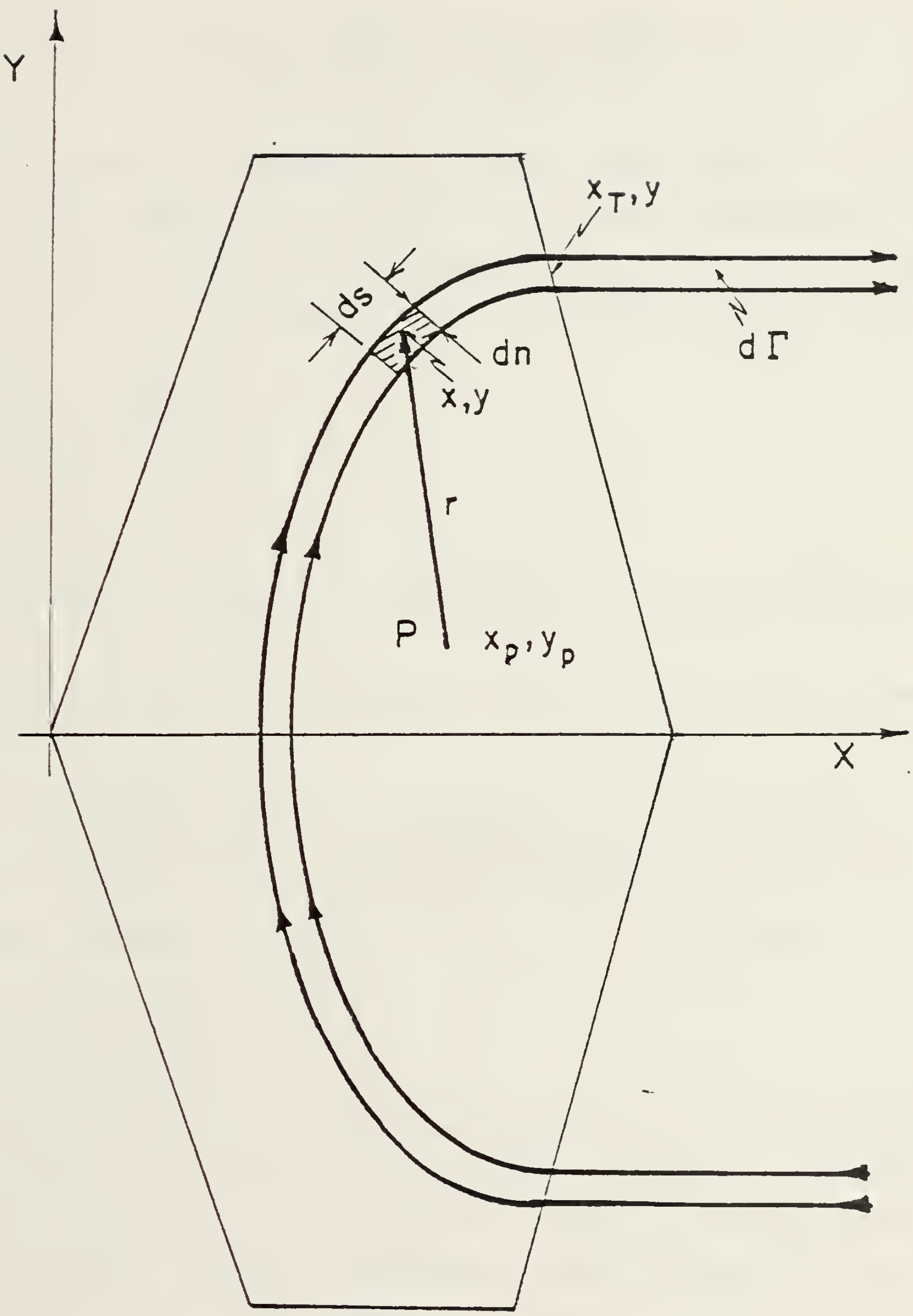

Fig. 5 Induced velocity in the x-y plane

Upon substituting expressions (3.2), (3.3), and (3.5) into (3.1), and integrating over the entire wing surface S (but not yet over the vortex sheet trailing behind the wing) the downwash due to the wing vortex is obtained.

$$\vec{W}_P' = \frac{\vec{k}}{4\pi}\int_S \frac{\left[(x-x_P)\left(\frac{\partial\Gamma}{\partial x}\right)+(y-y_P)\left(\frac{\partial\Gamma}{\partial y}\right)\right]}{\left[(x-x_P)^2+(y-y_P)^2\right]^{3/2}}\, dS \qquad (3.6)$$

The incremental downwash velocity $d\vec{w}_p''$ induced at point P due to the semi-infinite trailing vortex filament of strength $d\Gamma_T$ which leaves the wing at the point whose coordinates are x_T and y as shown in Fig. 5 is by the Biot Savart Law:

$$d\Gamma_T = -\left(\frac{d\Gamma_T}{dy}\right) dy$$

$$d\vec{W}_P'' = \frac{\vec{k}}{4\pi}\left(\frac{1}{y-y_P}\right)\left\{1-\frac{(x-x_P)}{\left[(x-x_P)^2+(y-y_P)^2\right]^{1/2}}\right\}\frac{d\Gamma_T}{dy}\, dy \qquad (3.7)$$

Integrating this over the entire span of the wing gives:

$$\vec{W}_P'' = \frac{\vec{k}}{4\pi}\int_{-s}^{s}\left(\frac{1}{y-y_P}\right)\left\{1-\frac{(x-x_P)}{\left[(x-x_P)^2+(y-y_P)^2\right]^{1/2}}\right\}\left(\frac{d\Gamma_T}{dy}\right) dy \qquad (3.8)$$

The total induced velocity at point P is the sum

$$\vec{W}_P = \vec{W}_P' + \vec{W}_P'' \qquad (3.9)$$

where $\vec{W}_P'$ and $\vec{W}_P''$ are evaluated respectively from equations (3.6) and (3.7). The first of these integrals represents the velocity induced by the vorticity over the actual wing surface, and the second integral represents the velocity induced by the vorticity trailing behind the wing.

The downwash integrals are now non-dimensionalized and the Cartesian coordinates x and y are transformed into dimensionless relative coordinates ξ and η defined as follows:

$$\xi = \frac{x-x_L}{c} = \text{relative chordwise coordinate} \tag{3.10}$$

$$\eta = \frac{y}{s} = \text{relative spanwise coordinate} \tag{3.11}$$

The following auxiliary definitions are used.

$$\bar{c} = \frac{S}{2s} = \text{average chord} \tag{3.12}$$

$$\mathcal{R} = \frac{2s}{\bar{c}} = \text{aspect ratio} \tag{3.13}$$

$$\Xi_L(\eta) = \frac{x_L}{\bar{c}} = \text{leading edge function} \tag{3.14}$$

$$\Xi_T(\eta) = \frac{x_T}{\bar{c}} = \text{trailing edge function} \tag{3.15}$$

$$\frac{c}{\bar{c}} = C(\eta) = \text{relative chord function} \tag{3.16}$$

$$C(\eta) = \Xi_T(\eta) - \Xi_L(\eta) \tag{3.17}$$

Note that if the parameter $\mathcal{R}$ is specified and if any two of the three functions involved in equation (3.17) are specified, this information completely defines the wing planform.

The circulation function Γ is non-dimensionalized as follows. The number four is inserted into these definitions, somewhat arbitrarily, because this reduces the final downwash integrals to a particularly convenient numerical form.

$$\Gamma^*(\xi,\eta) = \frac{\Gamma}{4V_\infty s} = \text{dimensionless circulation function along the wing surface} \tag{3.18}$$

$$\Gamma_T^*(\eta) = \frac{\Gamma_T}{4V_\infty s} = \text{dimensionless circulation function at and behind the trailing edge} \tag{3.19}$$

The induced velocities $\vec{w}_P'$ and $\vec{w}_P''$ are non-dimensionalized to corresponding downwash angles α_P' and α_P'' as shown below. Those angles are defined as positive if down from free stream velocity.

$\alpha_P' = -\dfrac{\vec{W}_P' \cdot \vec{k}}{V_\infty}$ = downwash angle point P induced by vorticity distribution over the wing (3.20)

$\alpha_P'' = -\dfrac{\vec{W}_P'' \cdot \vec{k}}{V_\infty}$ = downwash angle at point P induced by vorticity distribution behind the wing (3.21)

$\alpha_P = \alpha_P' + \alpha_P''$ = total downwash angle induced at point P by the complete vortex sheet system of the wing (3.22)

Some useful geometrical relationships are :

$$(x-x_P) = s\,\frac{2}{AR}\left[\left(X_L + C\xi\right) - \left(X_{LP} + C_P\,\xi_P\right)\right] \tag{3.23}$$

$$(y-y_P) = s\,(\eta - \eta_P) \tag{3.24}$$

Also at any point on the wing

$$\Gamma(x, y) = 4\,V_\infty\, s\,\Gamma^*(\xi, \eta) \tag{3.25}$$

Differentiating equation (3.25)

$$d\Gamma = \left(\frac{\partial \Gamma}{\partial x}\right)dx + \left(\frac{\partial \Gamma}{\partial y}\right)dy = 4V_\infty s\left[\left(\frac{\partial \Gamma^*}{\partial \xi}\right)d\xi + \left(\frac{\partial \Gamma^*}{\partial \eta}\right)d\eta\right] \tag{3.26}$$

Differentiating equations (3.24) and (3.25)

$$dx = s\frac{2}{A\!\!R}\left[C\,d\xi + (X_L' + C'\xi)\,d\eta - 0\right] \tag{3.27}$$

$$dy = s\,d\eta \tag{3.28}$$

The prime marks in equation (3.27) denotes differentiation with respect to y. Substituting equations (3.27) and (3.28) into equation (3.26) gives

$$\frac{d\Gamma}{s} = \left(\frac{\partial\Gamma}{\partial x}\right)\frac{2}{A\!\!R}\left[C\,d\xi + (X_L' + C'\xi)\,d\eta\right] + \frac{\partial\Gamma}{\partial y}\,dy$$

$$\frac{d\Gamma}{s} = 4V_\infty\left[\left(\frac{\partial\Gamma^*}{\partial\xi}\right)d\xi + \left(\frac{\partial\Gamma^*}{\partial\eta}\right)d\eta\right. \tag{3.29}$$

The coefficients of dz and dy must be separately equal on both sides of equation (3.29) since this relation must be satisfied for arbitrary values of dz and dy. Hence

$$\left(\frac{\partial\Gamma}{\partial x}\right)\frac{2}{A\!\!R}C = 4V_\infty\left(\frac{\partial\Gamma^*}{\partial\xi}\right) \tag{3.30}$$

$$\left(\frac{\partial\Gamma}{\partial x}\right)\frac{2}{A\!\!R}(X_L' + C'\xi) + \left(\frac{\partial\Gamma}{\partial y}\right) = 4V_\infty\left(\frac{\partial\Gamma^*}{\partial\eta}\right) \tag{3.31}$$

Solving those two equations for $\frac{\partial\Gamma}{\partial x}$ and $\frac{\partial\Gamma}{\partial y}$ gives

$$\left(\frac{\partial\Gamma}{\partial x}\right) = 4V_\infty\frac{A\!\!R}{2C}\left(\frac{\partial\Gamma^*}{\partial\xi}\right) \tag{3.32}$$

$$\left(\frac{\partial\Gamma}{\partial y}\right) = 4V_\infty\left[\left(\frac{\partial\Gamma^*}{\partial\eta}\right) - \left(\frac{X_L' + C'\xi}{C}\right)\left(\frac{\partial\Gamma^*}{\partial\xi}\right)\right. \tag{3.33}$$

The element of wing area in relative coordinates may be expressed in the following manner.

$$dS = dx)_\eta\,dy)_\xi = c\,d\xi \cdot s\,d\eta$$

$$dS = 2s^2\left(\frac{\bar{c}}{2s}\right)\left(\frac{c}{\bar{c}}\right)d\xi\,d\eta = s^2\frac{2}{A\!\!R}C(\eta)\,d\xi\,d\eta \tag{3.34}$$

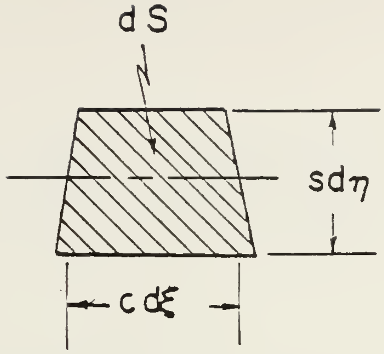

Fig. 6 Element of Wing Area

The various quantities of equations (3.6) and (3.8) are transformed into non-dimensional notation.

$$\left[(x-x_P)^2+(y-y_P)^2\right]^{3/2} = s^3\left\{\frac{4}{AR^2}\left[(\bar{X}_L+C\xi)-(\bar{X}_{LP}-C_P\xi_P)\right]^2+(\eta-\eta_P)^2\right\}^{3/2}$$

$$\left[(x-x_P)\frac{\partial\Gamma}{\partial x}+(y-y_P)\frac{\partial\Gamma}{\partial y}\right]dS = 4V_\infty s\left\{\frac{2}{AR}\left[(\bar{X}_L+C\xi)-(\bar{X}_{LP}+C_P\xi_P)\right]\right.$$

$$\left.\frac{AR}{2C}\left(\frac{\partial\Gamma^*}{\partial\xi}\right)+(\eta-\eta_P)\left[\left(\frac{\partial\Gamma^*}{\partial\eta}\right)-\left(\frac{\bar{X}_L'+C'\xi}{C}\right)\left(\frac{\partial\Gamma^*}{\partial\xi}\right)\right]\right\} s^2\,\frac{2}{AR}\,C\,d\xi\,d\eta$$

$$= 4V_\infty s^3\left\{\frac{2}{AR}\left[(\bar{X}_L+C\xi)-(\bar{X}_{LP}+C_P\xi_P)\right]\left(\frac{\partial\Gamma^*}{\partial\xi}\right)\right.$$

$$\left.+\ \frac{2}{AR}(\eta-\eta_P)\left[C\left(\frac{\partial\Gamma^*}{\partial\eta}\right)-(\bar{X}_L'+C'\xi)\left(\frac{\partial\Gamma^*}{\partial\xi}\right)\right]\right\}d\xi\,d\eta$$

$$= 4V_\infty s^3\,\frac{2}{AR}\left\{\left[(\bar{X}_L+C\xi)-(\bar{X}_{LP}+C_P\xi_P)-(\eta-\eta_P)(\bar{X}_L'+C'\xi)\right]\left(\frac{\partial\Gamma^*}{\partial\xi}\right)\right.$$

$$\left.+\ (\eta-\eta_P)\,C\left(\frac{\partial\Gamma^*}{\partial\eta}\right)\right\}d\xi\,d\eta \qquad (3.36)$$

$$(x_T-x_P) = s\,\frac{2}{AR}\left[\bar{X}_T-(\bar{X}_{LP}+C_P\xi_P)\right] \qquad (3.37)$$

Study of the algebraic structure of these relations shows that the final downwash integrals can be reduced to a relatively concise form by adopting the following auxiliary nomenclature. Let

$$F_1 = \frac{2\left[(X_L + C\xi) - (X_{LP} + C_P \xi_P) - (\eta - \eta_P)(X_L' + C'\xi)\right]}{\left\{\frac{4}{AR^2}\left[(X_L + C\xi) - (X_{LP} + C_P\xi_P)\right]^2 + (\eta - \eta_P)^2\right\}^{3/2}} \qquad (3.38)$$

$$F_2 = \frac{2(\eta - \eta_P)\,C}{\left\{\frac{4}{AR^2}\left[(X_L + C\xi) - (X_{LP} + C_P\xi_P)\right]^2 + (\eta - \eta_P)^2\right\}^{3/2}} \qquad (3.39)$$

$$F_3 = \left\{1 - \frac{\frac{2}{AR}\left[X_T - (X_{LP} + C_P\xi_P)\right]}{\left\{\frac{4}{AR^2}\left[X_T - (X_{LP} + C_P\xi_P)\right]^2 + (\eta - \eta_P)^2\right\}^{1/2}}\right\} \frac{1}{(\eta - \eta_P)} \qquad (3.40)$$

The downwash angle integrals equations (3.6) and (3.8) can now be expressed in the following form.

$$\alpha_P' = -\frac{1}{\pi AR}\int_{-1}^{1}\int_{0}^{1}\left\{F_1(\xi,\eta;\xi_P,\eta_P)\left(\frac{\partial\Gamma^*}{\partial\xi}\right) + F_2(\xi,\eta;\xi_P,\eta_P)\left(\frac{\partial\Gamma^*}{\partial\eta}\right)\right\} d\xi\, d\eta \qquad (3.41)$$

$$\alpha_P'' = -\frac{1}{\pi}\int_{-1}^{1} F_3(\eta;\xi_P,\eta_P)\left(\frac{d\Gamma_T^*}{d\eta}\right) d\eta \qquad (3.42)$$

D. THE CONDITION OF NO FLOW THROUGH THE WING

The shape of the mean wing surface near the x,y plane may be expressed by the function

$$z = z(x,y) \qquad (4.1)$$

A unit vector $\vec{n}_P$ normal to the above surface at the arbitrary point P on that surface, in terms of its Cartesian components, is given by the expression

$$\vec{n}_p = \frac{-\left(\frac{\partial z}{\partial x}\right)_p \vec{i} - \left(\frac{\partial z}{\partial y}\right)_p \vec{j} + (1)\vec{k}}{\left[\left(\frac{\partial z}{\partial x}\right)_p^2 + \left(\frac{\partial z}{\partial y}\right)_p^2 + 1\right]^{1/2}} \tag{4.2}$$

Since $\left(\frac{\partial z}{\partial x}\right)_p^2$ and $\left(\frac{\partial z}{\partial y}\right)_p^2$ are negligible in comparison with unity, equation (4.2) is adequately approximated by the linearized version

$$\vec{n}_p \simeq -\left(\frac{\partial z}{\partial x}\right)_p \vec{i} - \left(\frac{\partial z}{\partial y}\right)_p \vec{j} + (1)\vec{k} \tag{4.3}$$

The resultant velocity vector at point P may be closely approximated by the expression

$$\vec{V}_p = \vec{i}\, V_\infty \cos\alpha + \vec{k}\, V_\infty \sin\alpha - \vec{k}\, w_p \tag{4.4}$$

Equation (4.4) implies that the induced velocity w_p is taken as positive if down.

Since α is a small angle equation (4.4) may be linearized to

$$\vec{V}_p \simeq \vec{i}\, V_\infty + \vec{k}\left(V_\infty \alpha - w_p\right) \tag{4.5}$$

The condition of no flow through the wing now requires that

$$\vec{V}_p \cdot \vec{n} = 0 \tag{4.6}$$

thus

$$\left[\vec{i}\, V_\infty + \vec{k}\left(V_\infty \alpha - w_p\right)\right] \cdot \left[\vec{i}\left(\frac{\partial z}{\partial x}\right)_p - \vec{j}\left(\frac{\partial z}{\partial y}\right)_p + \vec{k}(1)\right] = 0 \tag{4.7}$$

or

$$-V_\infty \left(\frac{\partial z}{\partial x}\right)_p + \left(V_\infty \alpha - w_p\right) = 0 \tag{4.8}$$

hence

$$\left(\frac{\partial z}{\partial x}\right)_P = \alpha - \frac{w_P}{V_\infty} \qquad (4.9)$$

but

$$\frac{w_P}{V_\infty} = \alpha_P = \alpha_P' + \alpha_P'' \qquad (4.10)$$

therefore

$$\left(\frac{\partial z}{\partial x}\right)_P = \alpha - (\alpha_P' + \alpha_P'') \qquad (4.11)$$

The overall wing equation is now found by substituting the integrals of equations (3.41) and (3.42) into equation (4.11). The result is the basic wing equation.

$$\left(\frac{\partial z}{\partial x}\right)_P = \alpha + \frac{1}{\pi A\!\!R}\int_{-1}^{1}\int_{0}^{1}\left\{F_1\left(\frac{\partial \Gamma^*}{\partial \xi}\right) + F_2\left(\frac{\partial \Gamma^*}{\partial \eta}\right)\right\} d\xi\, d\eta$$

$$+ \frac{1}{\pi}\int_{-1}^{1} F_3\left(\frac{d\Gamma_T^*}{d\eta}\right) d\eta \qquad (4.12)$$

The influence functions F_1, F_2, and F_3 are as defined by equations (3.38), (3.39) and (3.40). These three functions depend only on the planform and aspect ratio of the wing. Theoretically, equation (4.12) must be satisfied at all points in the wing surface for an exact solution. However, an adequate approximate solution can frequently be obtained by satisfying equation (4.12) at a sufficient number of discrete control points suitably distributed over the wing surface.

E. REFERENCE AND ADDITIONAL LIFT DISTRIBUTIONS

Let the wing slope function $\left(\frac{\partial z}{\partial x}\right)_P$ be known at every point ξ_P, η_P of the wing. Also let the entire leading

edge of the wing lie exactly in the x-y plane and the trailing edge lie in the x-y plane at midspan. In general, however, z is not necessarily zero at other points along the trailing edge. Let $\Gamma_r^*(\xi,\eta)$ be a function which satisfies equation (4.12) when α has a particular value α_r. The subscript r stands for reference condition. Thus at the reference condition equation (4.13) becomes

$$\left(\frac{\partial z}{\partial x}\right)_P = \alpha_r + \frac{1}{\pi A\!\!R}\int_{-1}^{1}\int_{0}^{1}\left\{F_1\left(\frac{\partial \Gamma_r^*}{\partial \xi}\right) + F_2\left(\frac{\partial \Gamma_r^*}{\partial \eta}\right)\right\}d\xi\, d\eta$$

$$+ \frac{1}{\pi}\int_{-1}^{1} F_3\left(\frac{d\Gamma_{Tr}^*}{d\eta}\right)d\eta \tag{5.1}$$

The general solution $\Gamma^*(\xi,\eta)$ or equation (4.13) can now be written as the sum of the reference solution and an additional solution in the form

$$\Gamma^*(\xi,\eta) = \Gamma_r^*(\xi,\eta) + (\alpha-\alpha_r)\,\Gamma_a(\xi,\eta) \tag{5.2}$$

To demonstrate that equation (5.2) is valid, and to obtain the equation that governs the form of the additional solution function we proceed as follows. Substitute equation (5.2) into equation (4.12). Substract equation (5.1) from the result. Finally divide through by the factor $(\alpha-\alpha_r)$. The result is

$$0 = 1 + \frac{1}{\pi A\!\!R}\int_{-1}^{1}\int_{0}^{1}\left\{F_1\left(\frac{\partial \Gamma_a^*}{\partial \xi}\right) + F_2\left(\frac{\partial \Gamma_a^*}{\partial \eta}\right)\right\}d\xi\, d\eta$$

$$+ \frac{1}{\pi}\int_{-1}^{1} F_3\left(\frac{d\Gamma_{Ta}^*}{d\eta}\right)d\eta \tag{5.3}$$

Thus equation (5.1) governs the reference solution or the particular solution while equation (5.3) governs the additional solution. The specific value of α_r in equation (5.1) is fixed by the particular condition that

$$z = 0 \text{ at } \xi = +1 \; , \; \eta = 0 \tag{5.4}$$

given that

$$z = 0 \text{ at } \xi = 0 \text{ for all } \eta \tag{5.5}$$

The reference solution depends not only on the wing planform and aspect ratio but also on the wing slope function and the angle of attack α_r . However, the additional lift function involves only the wing planform and aspect ratio. It will be shown later that the wing lift curve slope depends only on the additional lift function Γ_α^* and is independent of the reference solution Γ_r^* . Therefore, the wing lift curve slope depends on wing planform and aspect ratio only, and is independent of the wing slope function.

To specify the design of a wing it is necessary to fix two planform functions $X_L(\eta)$ and $c(\eta)$, the aspect ratio AR , and the wing slope function $(\frac{\partial z}{\partial x})_P$ at all points ξ_P, η_P on the wing. For a wing of specified design, equation (5.1) must be solved for the initially unknown function $\Gamma_r^*(\xi, \eta)$ and equation (5.3) must be solved for the initially unknown function $\Gamma_\alpha^*(\xi, \eta)$.

In the case of equation (5.1) the above procedure can also be partially reversed, that is, the function $\Gamma_r^*(\xi, \eta)$ can be specified arbitrarily, within certain limits, and the equation can then be solved for $(\frac{\partial z}{\partial x})_P$ as the unknown. This reversed solution procedure had two advantages. Firstly, it means that the lift distribution over the wing in the reference condition can be stipulated independently and that the wing slope function can always be found such as to yield the desired reference lift distribution. Secondly, it happens that equation (5.1) is much easier to solve when the wing slope function is the unknown than when the reference lift distribution is the unknown because, in the latter case,

the unknown function is under the integral sign. If $(\frac{\partial z}{\partial x})_P$ is the unknown, equation (5.1) can be solved by direct integration; usually this must be performed numerically. Repeated integrations of equation (5.1) are required, a separate numerical integration being needed for each discrete point at which the wing slope function is being evaluated. In order to represent the wing slope function adequately it is necessary to choose a sufficient number of discrete points suitably distributed over the wing.

Unfortunately, the above reversed solution procedure cannot be applied to the additional lift distribution Γ_α^* as governed by equation (5.3). Of course, the equation can still be solved, at least approximately.

The choice of the reference angle of attack α_r which fixes the corresponding reference solution Γ_r^* is to a certain extent arbitrary, The most common choice is to set α_r equal to the angle of zero lift of the wing, that is to the angle which yields a wing lift coefficient of zero. For a symmetrical lift distribution the lift coefficient is

$$C_L = \int_0^1 c_\ell \left(\frac{c}{\bar{c}}\right) d\eta \qquad (5.6)$$

The restriction that the lift coefficient equal zero does not necessarily mean that the integrand is identically zero all along the span, just that the net area inder the curve sums to zero. Thus at $C_L = 0$, portions of the span may be exerting upward forces, while other portions are exerting downward forces. A wing that behaves in this way at $C_L = 0$ may be said to be aerodynamically twisted. Conversely a wing that at $C_L = 0$ yields $c_\ell \frac{c}{\bar{c}}$ identically zero for all values of η may be said to be aerodynamically untwisted; this however does not imply the absence of camber.

The reference condition for this analysis is that which corresponds to the angle of attack for which the wing lift coefficient equals zero. This particular lift distribution is also commonly labeled as the basic lift distribution.

F. ANGULAR COORDINATES

A further shift from the linear relative coordinates ξ, η to corresponding angular coordinates ϕ , θ , is now made. See Fig. 7. The basic conversion relations are

$$\xi = \tfrac{1}{2} (1 - \cos\phi) \tag{6.1}$$

$$\eta = \cos\theta \tag{6.2}$$

then

$$\frac{d\xi}{d\phi} = \tfrac{1}{2} \sin \phi \tag{6.3}$$

$$\frac{d\eta}{d\theta} = - \sin \theta \tag{6.4}$$

$$\frac{\partial}{\partial\xi} (\quad) = \frac{2}{\sin\phi} \frac{\partial}{\partial\phi} (\quad) \tag{6.5}$$

$$\frac{\partial}{\partial\eta} (\quad) = - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\quad) \tag{6.6}$$

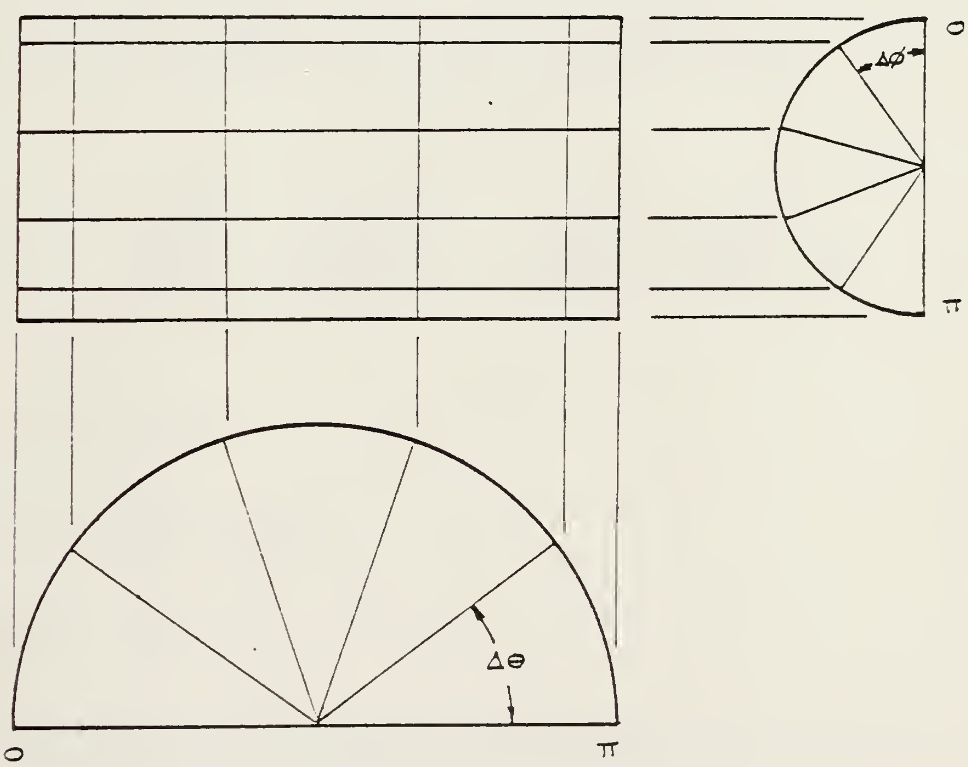

Fig. 7 Angular Coordinates

The downwash integrals can now be converted as follows

$$-\alpha_p' = \frac{1}{\pi A\!R}\int_{-1}^{1}\int_{0}^{1}\left\{F_1\left(\frac{\partial\Gamma^*}{\partial\xi}\right) + F_2\left(\frac{\partial\Gamma^*}{\partial\eta}\right)\right\} d\xi\, d\eta$$

$$-\alpha_p' = \frac{1}{\pi A\!R}\int_{\pi}^{0}\int_{0}^{\pi}\left\{F_1 \frac{2}{\sin\phi}\left(\frac{\partial\Gamma^*}{\partial\phi}\right) - F_2 \frac{1}{\sin\theta}\left(\frac{\partial\Gamma^*}{\partial\theta}\right)\right\}\frac{1}{2}\sin\phi\,(-\sin\theta)\, d\phi\, d\theta$$

$$-\alpha_p' = \frac{1}{\pi A\!R}\int_{0}^{\pi}\int_{0}^{\pi}\left\{F_1 \sin\theta\left(\frac{\partial\Gamma^*}{\partial\phi}\right) - F_2 \frac{\sin\phi}{2}\left(\frac{\partial\Gamma^*}{\partial\theta}\right)\right\} d\phi\, d\theta$$

$$-\alpha_p' = \frac{1}{\pi A\!R}\int_{0}^{\pi}\int_{0}^{\pi}\left\{G_1\left(\frac{\partial\Gamma^*}{\partial\phi}\right) + G_2\left(\frac{\partial\Gamma^*}{\partial\theta}\right)\right\} d\phi\, d\theta \qquad (6.7)$$

$$-\alpha_p'' = \frac{1}{\pi}\int_{-1}^{1} F_3\left(\frac{d\Gamma_T^*}{d\eta}\right) d\eta = \frac{1}{\pi}\int_{\pi}^{0} F_3\left(\frac{d\Gamma_T^*}{d\theta}\right) d\theta$$

$$-\alpha_p'' = -\frac{1}{\pi}\int_{0}^{\pi} F_3\left(\frac{d\Gamma_T^*}{d\theta}\right) d\theta = \frac{1}{\pi}\int_{0}^{\pi} G_3\left(\frac{\partial\Gamma_T^*}{\partial\theta}\right) d\theta \qquad (6.8)$$

where

$$G_1 = F_1 \sin\theta \qquad (6.9)$$

$$G_2 = -\frac{F_2}{2}\sin\phi \qquad (6.10)$$

$$G_3 = -F_3 \qquad (6.11)$$

Now by refering to the above relationship and to the previous

equations which define the functions F_1, F_2, and F_3 it is possible to summarize the final wing equations as the culmination of a series of systematic calculation steps. The following scheme of auxiliary notation is used.

$$\xi = \tfrac{1}{2} (1 - \cos \phi) \tag{6.12}$$

$$\xi_P = \tfrac{1}{2} (1 - \cos \phi_P) \tag{6.13}$$

$$\eta = \cos \theta \tag{6.14}$$

$$\eta_P = \cos \theta_P \tag{6.15}$$

$$h_1 = X_L(\theta) + C(\theta)\,\xi \tag{6.16}$$

$$h_{1P} = X_L(\theta_P) + C(\theta_P)\,\xi_P \tag{6.17}$$

$$h_2 = \frac{d}{d\theta}\left[X_L(\theta)\right] + \xi \frac{d}{d\theta}\left[C(\theta)\right] \tag{6.18}$$

Depending on planform, the derivatives $\frac{dX_L}{d\theta}$ and $\frac{dC}{d\theta}$ may be multi-valued at midspan, however for a symmetrical planform they are set to zero.

$$h_3 = 2 \sin \left\{ (h_1 - h_{1P}) - (\eta - \eta_P)\, h_2 \right\} \tag{6.19}$$

$$h_4 = \left\{ \left[\frac{2}{AR} (h_1 - h_{1P}) \right]^2 + (\eta - \eta_P)^2 \right\}^{3/2} \tag{6.20}$$

$$X_T = X_L + C \tag{6.21}$$

$$h_5 = \frac{2}{AR} (X_T - h_{1P}) \tag{6.22}$$

$$h_6 = \left\{ h_5^2 + (\eta - \eta_P)^2 \right\}^{1/2} \tag{6.23}$$

$$G_1 = \frac{h_3}{h_4} \tag{6.24}$$

$$G_2 = - \frac{(\eta - \eta_P)\, C \sin \phi}{h_4} \tag{6.25}$$

$$G_3 = -\frac{\left(1 - \frac{h_5}{h_6}\right)}{\eta - \eta_P} \tag{6.26}$$

The basic wing equations become the following

Reference lift

$$\left(\frac{\partial z}{\partial x}\right)_P = \alpha_r + \frac{1}{\pi AR}\int_0^{\pi}\int_0^{\pi}\left[G_1\left(\frac{\partial \Gamma_r^*}{\partial \phi}\right) + G_2\left(\frac{\partial \Gamma_r^*}{\partial \theta}\right)\right] d\phi\, d\theta$$

$$+ \frac{1}{\pi}\int_0^{\pi} G_3\left(\frac{d\Gamma_{Tr}}{d\theta}\right) d\theta \tag{6.27}$$

Additional lift

$$0 = 1 + \frac{1}{\pi AR}\int_0^{\pi}\int_0^{\pi}\left[G_1\left(\frac{\partial \Gamma_a^*}{\partial \phi}\right) + G_2\left(\frac{\partial \Gamma_a^*}{\partial \theta}\right)\right] d\phi\, d\theta$$

$$+ \frac{1}{\pi}\int_0^{\pi} G_3\left(\frac{d\Gamma_{Ta}^*}{d\theta}\right) d\theta \tag{6.28}$$

G. APPROXIMATE SERIES SOLUTION

The reference and additional distributions can be adequately represented by double series of the following form.

$$\Gamma_r^*(\phi, \theta) = \sum_{r=0}^{R+1} \sum_{s=1,3,5\ldots}^{S} B_{rs}\, H_r(\phi) \sin s\theta \tag{7.1}$$

$$\Gamma_a^*(\phi, \theta) = \sum_{r=0}^{R+1} \sum_{s=1,3,5\ldots}^{S} A_{rs}\, H_r(\phi) \sin s\theta \tag{7.2}$$

For a spanwise symmetrical lift distribution, only odd values of s are used. Ref. 1. There are (R+1) distinct functions H_r and ($\frac{S+1}{2}$) distinct sine functions in the above series. The total number of initially undetermined constants in each

of the above series is therefore

$$T = (R+1)\cdot\left(\frac{S+1}{2}\right) \tag{7.3}$$

If there are (R+1) chordwise stations and ($\frac{S+1}{2}$) spanwise stations over the semi-span, this establishes T control points. By satisfying each governing equation at each of the T control points we can solve for each set of T constants. Once the constants Brs and Ars are found, all aspects of the wing performance can be readily calculated including lift distributions, slope of the lift curve, induced drag, etc.

The character of the H functions is established as follows

$$\frac{\Delta P}{q_\infty} = \frac{2}{V_\infty}\left(\frac{\partial \Gamma}{\partial x}\right) \tag{7.4}$$

$$\frac{\partial \Gamma}{\partial x} = \frac{2\,A\!\!R\,V_\infty}{c}\left(\frac{\partial \Gamma^*}{\partial \xi}\right) \tag{7.5}$$

therefore

$$\frac{\Delta P}{q_\infty} = \left(\frac{4A\!\!R}{c}\right)\left(\frac{\partial \Gamma^*}{\partial \xi}\right) = \left(\frac{4A\!\!R}{c}\right)\frac{2}{\mathrm{SIN}\,\phi}\left(\frac{\partial \Gamma^*}{\partial \phi}\right) \tag{7.6}$$

For the additional lift, equation (7.2) is substituted into equation (7.6)

$$\left(\frac{\Delta P}{q_\infty}\right)_a = \left(\frac{4A\!\!R}{c}\right)\sum_{r=0}^{R+1}\sum_{S=1,3,5\cdots}^{S} Ars\left[\frac{2}{\mathrm{SIN}\,\phi}\left(\frac{dHr}{d\phi}\right)\right]\mathrm{SIN}\,S\theta \tag{7.7}$$

For the reference lift, equation (7.1) is substituted into equation (7.6)

$$\left(\frac{\Delta P}{q_\infty}\right)_r = \left(\frac{4A\!\!R}{c}\right)\sum_{r=0}^{R+1}\sum_{S=1,3,5\cdots}^{S} Brs\left[\frac{2}{\mathrm{SIN}\,\phi}\left(\frac{dHr}{d\phi}\right)\right]\mathrm{SIN}\,S\theta \tag{7.8}$$

It is now possible to make use of the results of thin

airfoil theory and set up the functions in equations (7.7) and (7.8) in an analogous form. Ref. 2. Thus for the special case of r = 0 we use

$$\frac{\partial H_o}{\partial \xi} = \frac{2}{\sin\phi} \frac{dH_o}{d\phi} = \frac{1+\cos\phi}{\sin\phi} \tag{7.9}$$

and for r ≠ 0 we use

$$\frac{\partial H_r}{\partial \xi} = \frac{2}{\sin\phi}\left(\frac{dH_r}{d\phi}\right) = \sin r\phi \tag{7.10}$$

Equations (7.9) and (7.10) can be integrated to yield

For r =0

$$H_o(\phi) = \frac{1}{2}(\phi + \sin\phi) \tag{7.11}$$

For r = 1

$$H_1(\phi) = \frac{1}{4}\left(\phi - \frac{\sin 2\phi}{2}\right) \tag{7.12}$$

For r ≥ 2

$$H_r(\phi) = \frac{1}{4}\left\{\frac{\sin(r-1)\phi}{r-1} - \frac{\sin(r+1)\phi}{r+1}\right\} \tag{7.13}$$

At the trailing edge, $\phi = \pi$, these relations yield

$$H_o(\pi) = \frac{\pi}{2} \tag{7.14}$$

$$H_1(\pi) = \frac{\pi}{4} \tag{7.15}$$

$$H_r(\pi) = 0 \tag{7.16}$$

By substituting these values into equations (7.1) and (7.2) the corresponding expressions for the trailing vorticity are obtained.

$$\Gamma^*_{Tr}(\theta) = \frac{\pi}{4}\sum_{s=1,3,5\cdots}^{S}(2B_{os} + B_{1s})\sin s\theta \tag{7.17}$$

$$\Gamma^{*}_{T\alpha}(\theta) = \frac{\pi}{4} \sum_{s=1,3,5\ldots}^{S} (2A_{os} + A_{1s}) \sin s\theta \qquad (7.18)$$

An ideal wing is one for which $\left(\frac{\Delta P}{q_\infty}\right)_r = \left(\frac{\Delta P}{q_\infty}\right)_\alpha = 0$ at the leading edge. This requires that Bos = 0 for all s. If the planform function is arbitrarily prescribed the wing should not be presumed to be an ideal wing and all terms in Bos should be included in the solution. The additional lift is always characterized by the singularity $\left(\frac{\Delta P}{q_\infty}\right) \rightarrow \pm\infty$ at the leading edge so that the terms in Aos must always be retained.

III DESCRIPTION OF NUMERICAL PROBLEM SOLUTION

A. CALCULATION SCHEME

The calculation scheme in this analysis uses two distinct meshes. There is a coarse mesh of control points, and a fine mesh for the purposes of accurate integration. It is desirable with respect to both meshes that the ratio of the number of spanwise stations over the semispan to the number of chordwise stations should be roughly equal to the aspect ratio divided by two. This gives roughly equal resolution in the spanwise and chordwise directions.

The total angular interval, 0 to π in each case is subdivided into equal sub-intervals and the calculation point located at the center of each sub-interval. The calculation points associated with the fine mesh must not coincide with any of the control points associated with the coarse mesh because this would cause the functions F_1 and F_2 to assume an indeterminate form. While this indeterminacy can be resolved it is best avoided. A good way to avoid this difficulty is to make the fine mesh an even submultiple of that of the coarse mesh. Then the control points will always lie along intersections of the boundaries of the area elements which comprise the fine mesh. See Fig. 8.

The double integration over the wing area reduces in this scheme to a summation over the points of the fine mesh. The number of terms in the series being equal to the number of control points.

If the pressure distribution is to be found for a given wing the coefficients of the reference and additional lift series are solved for first and then the pressure distribution can be calculated.

If an ideal wing camber is to be defined from a given chordwise pressure distribution, the wing equation is solved for $(\frac{\partial z}{\partial x})_P$. The terms in H_0 are dropped which eliminates the leading edge singularity, and an equal number of terms is added on to the end of the series. In other words, Bos is set equal to zero. The induced drag of the wing can be

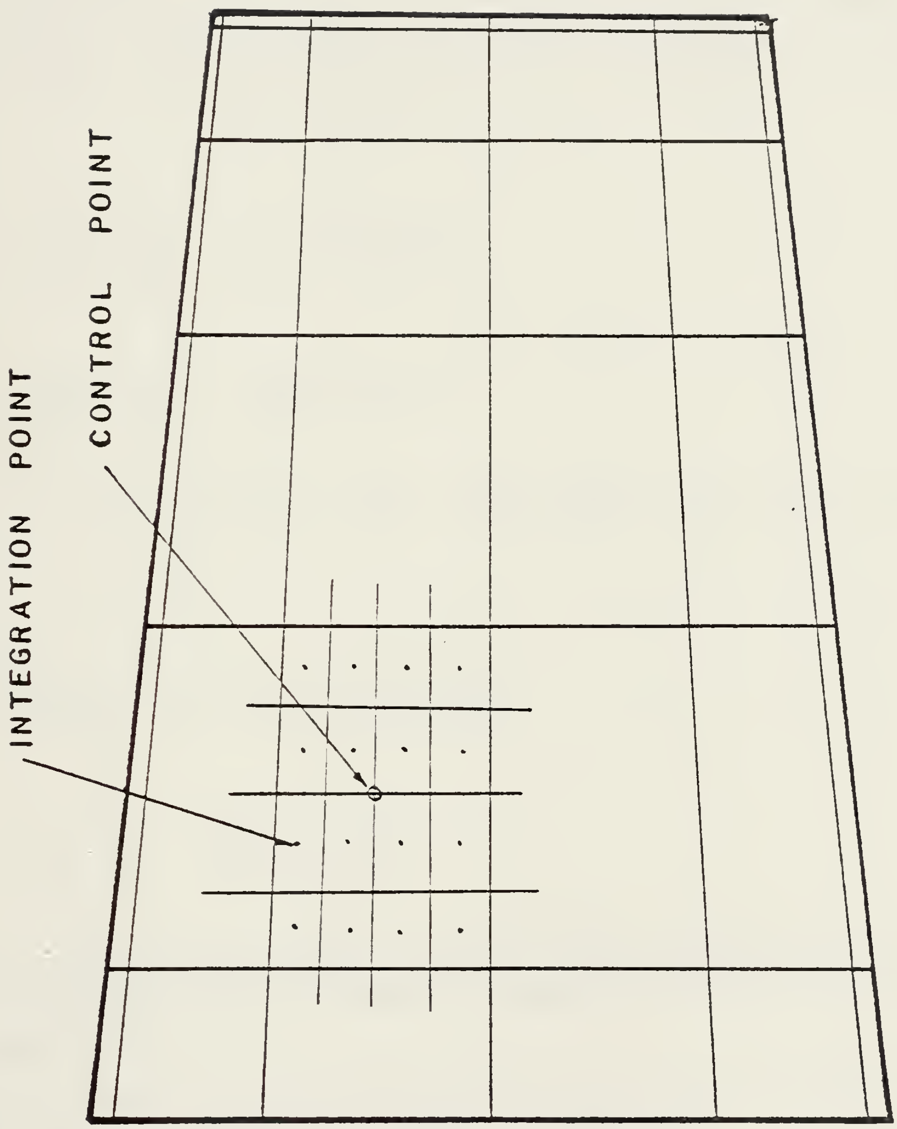

Fig. 8 Calculation Meshes

reduced to an absolute minimum if the terms in sin $s\theta$ are eliminated retaining only the term in sin θ . This defines an elliptical spanwise load distribution. These are desirable features to incorporate into the design of an ideal wing, and it also simplifies the numerical solution.

B. MATRIX FORMAT

By analogy with equations (7.7) and (7.8) the reference and additional lift distributions may be written in matrix format.

$$\left\{\left(\frac{\Delta P}{q_\infty}\right)_r\right\} = [Q]\{B\} \tag{8.1}$$

$$\left\{\left(\frac{\Delta P}{q_\infty}\right)_a\right\} = [Q]\{A\} \tag{8.2}$$

The matrix format of the basic wing equations (6.27) and (6.28) become

$$\left\{\frac{\partial z}{\partial x} - \alpha_r\right\} = [R]\{B\} \tag{8.3}$$

$$\{1\} = -[R]\{A\} \tag{8.4}$$

The matricies $[Q]$ and $[R]$ in the above equations are known T by T matricies whose elements depend only on wing planform and aspect ratio.

Solving equations (8.3) and (8.4) for the initially unknown constants B and A gives by matrix inversion

$$\{B\} = [R]^{-1}\left\{\frac{\partial z}{\partial x} - \alpha_r\right\} \tag{8.5}$$

$$\{A\} = -[R]^{-1}\{1\} \tag{8.6}$$

Equation (8.6) shows that the A constants depend only on wing planform and aspect ratio while the B constants in equation (8.5) also depend on the wing slope function.

The points used in the Q matrix need not be the same as the control points used in the R matrix, but may be any point on the wing. In this analysis the points for the Q matrix were picked as six stations across the semispan with eighteen chordwise stations at each spanwise station. This permits a good representation of the functions Γ and $(\frac{\Delta P}{q_\infty}) \frac{c}{C_L}$.

If the reference pressure distribution over the wing is specified, coefficients of the lift distribution series can be found, by equation (8.7). Then the wing equation (8.8) can be solved by direct numerical integration.

$$\left\{ \left(\frac{\Delta P}{q_\infty} \right)_r \right\} = \left[\phi \right] \left\{ B \right\} \tag{8.7}$$

$$\frac{\partial z}{\partial x} - \alpha_r = \frac{d\phi \, d\theta}{\pi A\!\!\!R} \sum_0^{\pi} \sum_0^{\pi} \Big[G_1 \sum^{R} B_r \frac{\sin r\phi \, \sin \phi}{2} \sin \theta + G_2 \sum^{R} B_r H_r \cos \theta \Big] + \frac{1}{4} \sum_0^{\pi} G_3 B_1 \cos \theta \, d\theta \tag{8.8}$$

IV RESULTS

A. SOLVING FOR THE WING PRESSURE DISTRIBUTION

The numerical results of the pressure distribution solution were not satisfactory. The pressure distributions and circulation functions calculated by the computer program were not smooth or in accordance with [Ref. 1] and [Ref. 10] . It is believed this is a result of difficulties with the first chordwise term, Bos. The term by term downwash is plotted in Fig. 9 , at a spanwise station near midspan for a wing of elliptical planform and an aspect ratio of 20 using 5 chordwise stations and 10 stations over the span. The integration mesh was four times finer than the control point mesh. The first term should yeild uniform downwash. the rest of the terms are of the general nature expected.

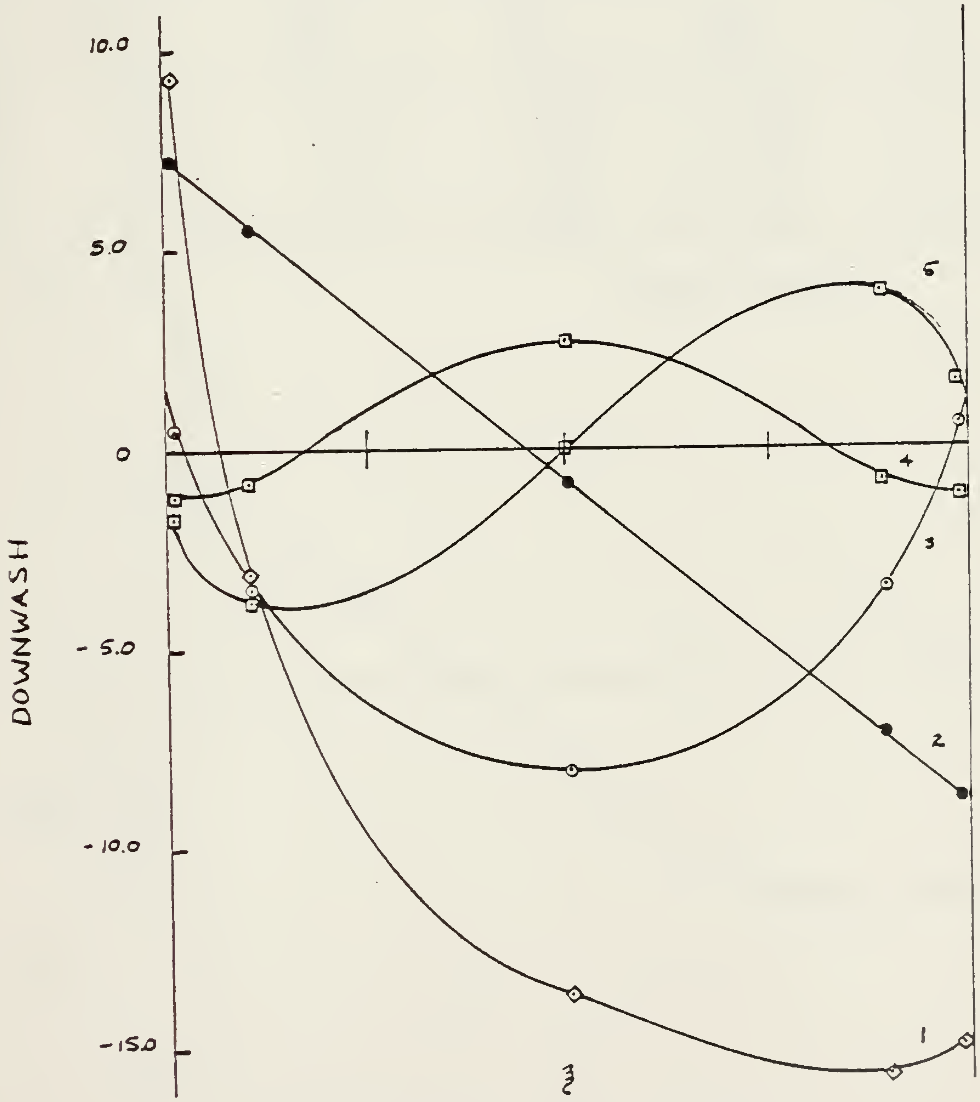

Fig. 9 Chordwise Downwash

B. SOLVING FOR THE WING CAMBER

The numerical results of calculating the camber of an ideal wing given the desired chordwise pressure distribution were satisfactory. Fig. 10 compares the results of $\frac{\partial z}{\partial x} - \alpha_r$ for a wing of aspect ratio 20 and rectangular planform using a 15 term lift distribution series at 5 chordwise points near mid-span, with the two dimensional results of [Ref. 10.] It is felt the differences at the leading edge can be resolved by using more terms in the lift distributions series and more integration points. The depicted pressure distribution was imposed at all spanwise stations. The results are normalized to a lift coefficient of one. Table 1 shows the results at various spanwise stations.

Table 1

AR=20 15 term lift distribution series rectangular planform

η	0.99	0.89	0.71	0.45	0.16
ξ	$\frac{\partial z}{\partial x} - \alpha_r$	$\frac{\partial z}{\partial x} - \alpha_r$	$\frac{\partial z}{\partial x} - \alpha_r$	$\frac{\partial z}{\partial x} - \alpha_r$	$\frac{\partial z}{\partial x} - \alpha_r$
0.025	0.0921	0.1400	0.1669	0.1782	0.1817
0.206	0.0292	0.0561	0.0789	0.0883	0.0913
0.500	-0.0448	-0.0781	-0.0965	-0.1038	-0.1611
0.794	-0.0967	-0.1552	-0.1951	-0.2152	-0.2297
0.976	-0.0998	-0.1456	-0.1884	-0.2142	-0.2256

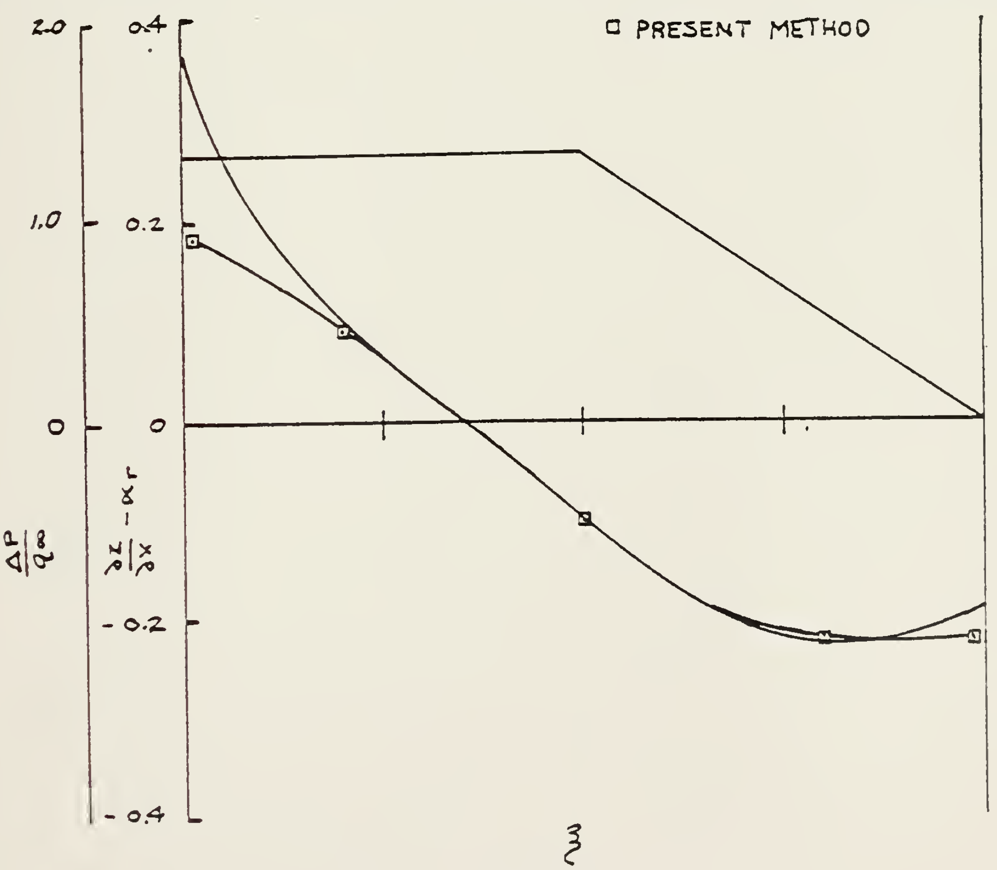

Fig. 10 Camber Line

V. CONCLUSIONS

The numerical implementation of the theory needs additional work to achieve the correct computer solutions for the pressure distribution solution. The areas of greatest interest are the assumed series solutions for the additional and reference lift, especially the first chordwise term, Bos. The camber line solution is consistent with two dimensional theory near midspan of a high aspect ratio wing and allows the camber line to be determined at various spanwise stations for any aspect ratio. It is felt that this theory has merit and deserves further study to correct the numerical solution of the pressure distribution.

APPENDIX A. DERIVATION OF SUBROUTINES

A. SUBROUTINES FOR THE PRESSURE DISTRIBUTION SOLUTION

QMAT calcualtes the Q matrix of equations (8.1) and (8.2)

$$\left(\frac{\Delta P}{q_\infty}\right)_a = \frac{4AR}{C} \sum^{R+1} \sum^{S} A_{rs} \left[\frac{2}{\sin\phi} \frac{dH_r}{d\phi}\right] \sin s\theta \tag{7.7}$$

$$\left(\frac{\Delta P}{q_\infty}\right)_a = \sum^{R+1} \sum^{S} A_{rs} \frac{4AR}{C} \left\{\left[\frac{2}{\sin\phi} \frac{dH_r}{d\phi}\right] \sin s\theta\right. \tag{A.1}$$

$$HH1 = \frac{2}{\sin\phi} \frac{dH_r}{d\phi} \tag{A.2}$$

$$S1 = \sin s\theta \tag{A.3}$$

$$\Phi(r,s) = \frac{4AR}{C} \cdot HH1 \cdot S1 \tag{A.4}$$

RMAT calcualtes the R matrix of equations (8.3) and (8.4)

$$\left(\frac{\partial z}{\partial x}\right)_P - \alpha_r = \frac{1}{\pi AR} \int_0^\pi \int_0^\pi \left(G_1 \frac{\partial \Gamma_r^*}{\partial \phi} + G_2 \frac{\partial \Gamma_r^*}{\partial \theta}\right) d\phi\, d\theta$$

$$+ \frac{1}{\pi} \int_0^\pi G_3 \left(\frac{d\Gamma_{Tr}^*}{d\theta}\right) d\theta \tag{6.27}$$

$$\frac{\partial \Gamma_r^*}{\partial \phi} = \sum^{R+1} \sum^{S} B_{rs} \frac{\partial}{\partial \phi}(H_r) \sin s\theta \tag{A.5}$$

$$\frac{\partial \Gamma_r^*}{\partial \theta} = \sum^{R+1} \sum^{S} B_{rs} H_r\, s \cos s\theta \tag{A.6}$$

$$\frac{d\Gamma_{Tr}^*}{d\theta} = \frac{\pi}{4} \sum^{S} (2B_{0s} + B_{1s})\, s \cos s\theta \tag{A.7}$$

$$\left(\frac{\partial z}{\partial x}\right)_p - \alpha_r = \sum^{RM}\sum^{S} B_{rs}\left\{\sum_0^{\pi}\sum_0^{\pi}\frac{d\phi\, d\theta}{\pi AR}\left[G_1\frac{\partial}{\partial\phi}(H_r)\sin s\theta\right.\right.$$

$$\left.\left.+\, G_2\, H_r\, s\cos s\theta\right]\right\} + \frac{1}{4}\sum^{S}(2B_{0s}+B_s)\sum_0^{\pi}G_3\sin s\theta\, d\theta \quad (A.8)$$

$$HH1 = \frac{\partial}{\partial\phi}(H_r) \quad (A.9)$$

$$S1 = \sin s\theta \quad (A.10)$$

$$HH2 = H_r \quad (A.11)$$

$$S2 = s\cos s\theta \quad (A.12)$$

$$R(r,s) = \sum_0^{\pi}\sum_0^{\pi}\frac{d\phi\, d\theta}{\pi AR}\left[G_1\cdot HH1\cdot S1 + G_2\cdot HH2\cdot S2\right]$$

$$+\sum_0^{\pi} G_3\cdot S2\cdot d\theta \quad (A.13)$$

B. SUBROUTINES FOR THE CAMBER LINE SOLUTION

QMAT calculates the Q matrix of equation (8.1) which has been simplified with a single term in theta, an elliptical spanwise lift distribution, and the first chordwise term is dropped.

$$\left(\frac{\Delta P}{q_\infty}\right)_r = \frac{4AR}{c}\sum^{R} B_r\left[\frac{2}{\sin\phi}\,\frac{dH_r}{d\phi}\right]\sin\theta \quad (A.14)$$

$$HH1 = \frac{2}{\sin\phi}\,\frac{dH_r}{d\phi} \quad (A.15)$$

$$\Phi(r) = \frac{4 AR}{c} \cdot HH1 \cdot S1 \tag{A.16}$$

CAMBER is a numerical integration of equation (6.27) simplified for an ideal wing.

$$\frac{\partial z}{\partial x} - \alpha r = \frac{d\phi\, d\theta}{\pi AR} \sum_0^{\pi} \sum_0^{\pi} \left[G_1 \sum^{R} B_r \frac{\sin r\phi \sin\phi}{2} \sin\theta \right.$$

$$\left. + G_2 \sum^{R} B_r H_r \cos\theta \right] + \frac{1}{4} \sum_0^{\pi} G_3 B_1 \cos\theta \, d\theta \tag{A.17}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C     THE PURPOSE OF THIS PROGRAM IS TO SOLVE FOR THE CHARACTERISTICS C
C     OF A WING OF GIVEN PLANFORM USING LIFTING SURFACE THEORY .  THE C
C     WING IS DIVIDED INTO ELEMENTS IN THE CHORDWISE AND SPANWISE     C
C     DIRECTION AND THE VORTICITY IS INTEGERATED OVER THE WING USING  C
C     A SERIES REPRESENTATION .                                       C
C     RP1 = THE NUMBER OF CHORDWISE ELEMENTS                          C
C     SP1 = THE NUMBER OF SPANWISE ELEMENTS (MUST BE EVEN)            C
C     MESH = THE NUMBER OF TIMES FINER THE INTEGERATION MESH IS THAN  C
C     THE ELEMENT MESH (MUST BE EVEN)                                 C
C     DZDXMA = THE CAMBER LINE SLOPE MINUS THE REFERENCE ANGLE OF     C
C     ATTACK                                                          C
C     DIMENSION DTATEMENTS: SP1D2= SP1/2   T=SP1D2*RP1  W=T**2+3*T    C
C         DIMENSION R(T,T),RM(T,T),Q(T,T)                             C
C         DIMENSION A(T),B(T),DZDXMA(T)                               C
C         DIMENSION WK(W)                                             C
C         DIMENSION RR(T,T)                                           C
C         DIMENSION Q(T,T)                                            C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER SP1,RP1,S,SP1D2,T
      INTEGER*4 ITB1(12)/12*0/,ITB2(12)/12*0/
      INTEGER*4 ITB3(12)/12*0/
      REAL*4 RTB1(28)/28*0.0/
      REAL*4 RTB2(28)/28*0.0/
      REAL*4 RTB3(28)/28*0.0/
      REAL*8 TITLE1(12)
      EQUIVALENCE (TITLE1,RTB1(5))
      REAL*8 TITLE2(12)
      EQUIVALENCE (TITLE2,RTB2(5))
      REAL*8 TITLE3(12)
      EQUIVALENCE (TITLE3,RTB3(5))
      DIMENSION R(25,25),RM(25,25),Q(25,25),RR(25,25)
      DIMENSION A(25),B(25),DZDXMA(25)
      DIMENSION WK(700)
      DIMENSION CPA(108),RPC(108),PHIP(18),THETAP(6),XI(18),ETA(6)
      DIMENSION RPC1(18),RPC2(18),RPC3(18),RPC4(18),RPC5(18),RPC6(18)
      DIMENSION CPA1(18),CPA2(18),CPA3(18),CPA4(18),CPA5(18),CPA6(18)
      DIMENSION GAMMA(6,18),GAMMA1(18),GAMMA2(18),GAMMA3(18),GAMMA4(18)
      DIMENSION GAMMA5(18),GAMMA6(18)
      DIMENSION XIT(6),GAMMAT(6)
      PI=3.1415927
      DATA ITB1/1,0,6,8,1,1,0,2,0,0,0,1/
      DATA ITB2/1,0,6,8,1,1,0,2,1,0,0,1/
      DATA ITB3/1,0,6,8,1,1,0,0,0,0,0,1/
```

```
      DATA XIT/1.0,1.0,1.0,1.0,1.0,1.0/
C
C     INPUT DATA
C
      READ (5,100) SP1,RP1,MESH
      READ (5,200) AR
      READ(5,600) TITLE1
      READ(5,600) TITLE2
      READ(5,600) TITLE3
      S=SP1-1
      SP1D2=SP1/2
      T=RP1*SP1D2
      READ (5,201)(DZDXMA(I),I=1,T)
C
C     PRINT INPUT DATA
C
      WRITE(6,101)
      WRITE(6,102) SP1,RP1,MESH
      WRITE(6,103) AR
      WRITE(6,104)
      WRITE(6,301)(DZDXMA(I),I=1,T)
C
C     CCMPUTE THE Q MATRIX
C
      CALL QMAT(SP1,RP1,AR,Q,PHIP,THETAP)
      WRITE(6,300)
      WRITE(6,301)((Q(I,J),J=1,T),I=1,T)
C
C     CCMPUTE THE R MATRIX
C
      CALL RMAT(SP1,RP1,MESH,AR,R)
      WRITE(6,302)
      WRITE(6,301)((R(I,J),J=1,T),I=1,T)
C
C     SOLVE FOR THE A VECTOR
C
      DO 1 L=1,T
      DO 1 M=1,T
      RR(L,M)=R(L,M)
    1 RM(L,M)=-R(L,M)
C
      DO 2 K=1,T
    2 A(K)=1.0
C
      IDGT=5
      CALL LEQT2F(RM,1,T,T,A,IDGT,WK,IER)
      WRITE(6,303)
      WRITE(6,301)(A(I),I=1,T)
```

```
      WRITE(6,500) IDGT
C
C     SOLVE FOR THE B VECTOR
C
      DO 3 I=1,T
    3 B(I)=DZDXMA(I)
C
      CALL DGELG(B,R,T,1,1.D-4,IER2)
      WRITE(6,306)
      WRITE(6,301)(B(I),I=1,T)
      WRITE(6,501) IER2
C
C     SOLVE FOR THE ADDITIONAL PRESSURE DISTRIBUTION
C
      CALL DGMPRD(Q,A,CPA,T,T,1)
      WRITE(6,305)
      WRITE(6,301)(CPA(I),I=1,T)
C
C     SOLVE FOR THE REFERENCE PRESSURE DISTRIBUTION
C
      CALL DGMPRD(Q,B,RPC,T,T,1)
      WRITE(6,307)
      WRITE(6,301)(RPC(I),I=1,T)
C
      DCLADA=(2.0*A(1)+A(SP1D2+1))*AR/4.0*PI**2
      WRITE(6,402) DCLADA
      CLR=(2.0*B(1)+B(SP1D2))*AR/4.0*PI**2
      WRITE(6,401) CLR
C
      DO 30 I=1,RP1
      CPA1(I)=CPA(I*5-4)/DCLADA
      CPA2(I)=CPA(I*5-3)/DCLADA
      CPA3(I)=CPA(I*5-2)/DCLADA
      CPA4(I)=CPA(I*5-1)/DCLADA
      CPA5(I)=CPA(I*5)/DCLADA
   30 CONTINUE
      IF(CLR .LT. 1.0D-4) GO TO 35
      DO 34 I=1,T
      RPC1(I)=RPC(I*5-4)/CLR
      RPC2(I)=RPC(I*5-3)/CLR
      RPC3(I)=RPC(I*5-2)/CLR
      RPC4(I)=RPC(I*5-1)/CLR
      RPC5(I)=RPC(I*5)/CLR
   34 CONTINUE
      GO TO 36
   35 CONTINUE
      DO 36 I=1,T
      RPC1(I)=RPC(I*5-4)
```

```
      RPC2(I)=RPC(I*5-3)
      RPC3(I)=RPC(I*5-2)
      RPC4(I)=RPC(I*5-1)
      RPC5(I)=RPC(I*5)
   36 CCNTINUE
C
C
C     SOLVE FOR THE GAMMA DISTRIBUTION
C
      DC 40 I=1,SP1D2
      DO 40 J=1,RP1
      GAMMA(I,J)=0.0
      DO 41 K=1,RP1
      DO 41 L=1,S,2
      GAM=A(K*SP1D2-SP1D2+(L+1)/2)*DSIN(L*THETAP(I))
      IF (K-2) 42,43,44
   42 HR=0.5*(PHIP(J)+DSIN(PHIP(J)))
      GO TO 45
   43 HR=0.25*(PHIP(J)-DSIN(2.0*PHIP(J))/2.0)
      GO TO 45
   44 HR=0.25*((DSIN((K-2)*PHIP(J))/(FLOAT(K)-2.0))-(DSIN(K*PHIP(J))/FLO
     1AT(K)))
   45 CCNTINUE
      GAM=GAM*HR
      GAMMA(I,J)=GAM+GAMMA(I,J)
   41 CONTINUE
   40 CCNTINUE
C
      DO 46 I=1,RP1
      GAMMA1(I)=GAMMA(1,I)
      GAMMA2(I)=GAMMA(2,I)
      GAMMA3(I)=GAMMA(3,I)
      GAMMA4(I)=GAMMA(4,I)
      GAMMA5(I)=GAMMA(5,I)
   46 CONTINUE
      DO 47 I=1,SP1D2
      GAMMAT(I)=0.0
      DO 47 L=1,S,2
      GAMMAA=(2.0*A((L+1)/2)+A(SP1D2+(L+1)/2))*DSIN(L*THETAP(I))*PI/4.0
      GAMMAT(I)=GAMMAA+GAMMAT(I)
   47 CCNTINUE
C
      DO 31 I=1,RP1
   31 XI(I)=0.5*(1.0-DCOS(PHIP(I)))
      WRITE(6,403)
      WRITE(6,308)(XI(I),I=1,18)
C
C     PLOT RESULTS
```

```
C
      RTB1(2)=0.5
      CALL DRAWP(RP1,XI,CPA5,ITB1,RTB1)
      ITB1(1)=2
      ITB1(2)=1
      CALL DRAWP(RP1,XI,CPA4,ITB1,RTB1)
      ITB1(2)=2
      CALL DRAWP(RP1,XI,CPA3,ITB1,RTB1)
      ITB1(2)=3
      CALL DRAWP(RP1,XI,CPA3,ITB1,RTB1)
      ITB1(2)=4
      CALL DRAWP(RP1,XI,CPA2,ITB1,RTB1)
      ITB1(1)=3
      ITB1(2)=5
      CALL DRAWP(RP1,XI,CPA1,ITB1,RTB1)
      RTB2(2)=2.0
      CALL DRAWP( T,XI,RPC1,ITB2,RTB2)
      ITB2(1)=2
      ITB2(2)=1
      CALL DRAWP( T,XI,RPC2,ITB2,RTB2)
      ITB2(2)=2
      CALL DRAWP( T,XI,RPC3,ITB2,RTB2)
      ITB2(2)=3
      CALL DRAWP( T,XI,RPC4,ITB2,RTB2)
      ITB2(2)=4
      ITB2(1)=3
      CALL DRAWP( T,XI,RPC5,ITB2,RTB2)
      CALL DRAWP(RP1,XI,GAMMA5,ITB3,RTB3)
      ITB3(1)=2
      ITB3(2)=1
      CALL DRAWP(RP1,XI,GAMMA4,ITB3,RTB3)
      ITB3(2)=2
      CALL DRAWP(RP1,XI,GAMMA3,ITB3,RTB3)
      ITB3(2)=3
      CALL DRAWP(RP1,XI,GAMMA2,ITB3,RTB3)
      ITB3(2)=4
      CALL DRAWP(RP1,XI,GAMMA1,ITB3,RTB3)
      ITB3(1)=3
      ITB3(2)=5
      CALL DRAWP(RP1,XIT,GAMMAT,ITB3,RTB3)
C
  100 FORMAT (3I10)
  101 FORMAT('1',5X,'LIFTING SURFACE SOLUTION FOR A WING')
  102 FORMAT('0',5X,'NO. OF SPANWISE ELEMENTS=',I2,2X,'NO. OF CHORDWISE
     1ELEMENTS=',I2,2X,'INTEGERATION MESH SIZE=',I2)
  103 FORMAT('0',5X,'WING ASPECT RATIO=',F5.2)
  104 FORMAT('0',5X,'THE CAMBER LINE SLOPE DZ/DX MINUS THE REFERENCE ANG
     1LE OF ATTACK ALPHA R VECTOR')
```

```
105 FORMAT(2F10.2)
106 FORMAT('0',5X,'WING ANGLE OF ATTACK = ',F5.2,2X,'REFERENCE ANGLE O
   1F ATTACK =',F5.2)
200 FORMAT (F10.2)
201 FORMAT(8F10.8)
300 FORMAT('0',5X,'THE Q MATRIX')
301 FORMAT('0',5(2X,F10.4))
302 FORMAT('0',5X,'THE R MATRIX')
303 FORMAT('0',5X,'THE A VECTOR')
305 FORMAT('0',5X,'THE ADDITIONAL LIFT VECTOR')
306 FORMAT('0',5X,'THE B VECTOR')
307 FORMAT('0',5X,'THE REFERENCE LIFT VECTOR')
308 FORMAT( 6(2X,F10.5))
309 FORMAT( 10(2X,F10.4))
400 FORMAT('0',5X,'THE LIFT DISTRIBUTION IS')
401 FORMAT('0',5X,'THE WING REFERENCE CL IS',2X,F10.4)
402 FORMAT('0',5X,'TNE WING LIFT CURVE SLOPE IS',2X,F10.4)
403 FORMAT('0',2X,'XI-CHORDWISE RELATIVE COORDINATES'/)
404 FORMAT('0',2X,'ETA-SPANWISE RELATIVE COORDINATES'/)
500 FORMAT('0',5X,'ERROR CODE FOR THE A VECTOR ',I4)
501 FORMAT('0',5X,'ERROR CODE FOR THE B VECTOR ',I4)
600 FORMAT(6A8)
601 FORMAT('0',5E25.7)
700 FORMAT('0',2X,'GAMMA DISTRIBUTION')
701 FORMAT('0',2X,'DOWNWASH ANGLES')
702 FORMAT('0',2X,'DOWNWASH TERMS')
    STOP
    END


C   PLANFORM FUNCTIONS


    FUNCTION XL(THETA)
    IMPLICIT REAL*8 (A-H,O-Z)
    XL=0.6366198*(1.0-DSIN(THETA))
    RETURN
    END


    FUNCTION C(THETA)
    IMPLICIT REAL*8 (A-H,O-Z)
```

```
      C=1.2732395*DSIN(THETA)
      RETURN
      END


      FUNCTION DXL(THETA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DXL=-0.6366198*DCOS(THETA)
      RETURN
      END


      FUNCTION DC(THETA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DC=1.2732395*DCOS(THETA)
      RETURN
      END


      SUBROUTINE RMAT(SP1,RP1,MESH,AR,RR)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER RP1,SP1,SP1D2,S,T
      DIMENSION RR(25,25)
      PI=3.1415927
      S=SP1-1
      SP1D2=SP1/2
      MCRI=RP1*MESH
      MSSJ=SP1*MESH
      T=SP1D2*RP1
      DELPHI=PI/FLOAT(MCRI)
      DTHETA=PI/FLOAT(MSSJ)
C
      DO 4 I=1,T
      DO 4 J=1,T
    4 RR(I,J)=0.0
C
      DO 1 M=1,RP1
      DO 1 N=1,SP1D2
      PHIP=PI/RP1*(M-0.5)
      THETAP=PI/SP1*(N-0.5)
      XIP=0.5*(1.0-DCOS(PHIP))
```

```
      ETAP=DCOS(THETAP)
      H1P=XL(THETAP)+C(THETAP)*XIP
      MN=M*SP1D2-SP1D2+N
C
      DO 2 K=1,RP1
      DO 2 L=1,S,2
      KL=K*SP1D2-SP1D2+(L+1)/2
C
      DO 3 I=1,MCRI
      DO 3 J=1,MSSJ
      PHI=PI/FLOAT(MCRI)*(FLOAT(I)-0.5)
      XI=0.5*(1.0-DCOS(PHI))
      THETA=PI/FLOAT(MSSJ)*(FLOAT(J)-0.5)
      ETA=DCOS(THETA)
      DETA=ETA-ETAP
      H1=XL(THETA)+C(THETA)*XI
      H2=DXL(THETA)+(XI*DC(THETA))
      DH1=H1-H1P
      H3=2.0*DSIN(THETA)*(DH1-(DETA*H2))
      H4=((2.0/AR*DH1)**2+DETA**2)**1.5
      G1=H3/H4
      G2=-DETA*C(THETA)*DSIN(PHI)/H4
      S1=DSIN(L*THETA)
      S2=L*DCOS(L*THETA)
C
      IF (K-1) 5,5,6
    5 HH1=0.5*(1.0+DCOS(PHI))
      GC TO 14
    6 HH1=0.5*(DSIN((K-1)*PHI)*DSIN(PHI))
   14 CONTINUE
      R1=G1*HH1*S1
C
      IF (K-2) 7,8,9
    7 HH2=0.5*(PHI+DSIN(PHI))
      GO TO 13
    8 HH2=0.25*(PHI-(DSIN(2.0*PHI)/2.0))
      GO TO 13
    9 HH2=0.25*((DSIN((K-2)*PHI)/(FLOAT(K)-2.0))-(DSIN(K*PHI)/FLOAT(K)))
   13 CONTINUE
      R2=G2*HH2*S2
      RR1=(DELPHI*DTHETA)/(PI*AR)*(R1+R2)
C
      IF (I.LT.MCRI) GO TO 12
      H5=2.0/AR*(XL(THETA)+C(THETA)-H1P)
      H6=(H5**2+DETA**2)**0.5
      G3=-(1.0-H5/H6)/DETA
C
      IF (K-2) 10,11,12
```

```
   10 RR2=G3*0.5*S2*DTHETA
      GO TO 15
   11 RR2=G3*0.25*S2*DTHETA
      GO TO 15
   12 RR2=0.0
   15 CONTINUE
      RR(MN,KL)=RR1+RR2+RR(MN,KL)
    3 CONTINUE
    2 CONTINUE
    1 CONTINUE
      RETURN
      END



      SUBROUTINE QMAT(SP1,RP1,AR,Q,PHIP,THETAP)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      INTEGER RP1,SP1,SP1D2,S,T
      DIMENSION Q(25,25)
      DIMENSION PHIP(18),THETAP(6)
      S=SP1-1
      SP1D2=SP1/2
      PI=3.1415927
C
      DC 1 M=1,RP1
      DO 1 N=1,SP1D2
      PHIP(M)=PI/RP1*(M-0.5)
      THETAP(N)=PI/SP1*(N-0.5)
      MN=M*SP1D2-SP1D2+N
C
      DC 2 K=1,RP1
      DO 2 L=1,S,2
      KL=K*SP1D2-SP1D2+(L+1)/2
C
      IF(K-1) 3,3,4
    3 HH1=(1.0+DCOS(PHIP(M)))/DSIN(PHIP(M))
      GC TO 5
    4 HH1=DSIN((K-1)*PHIP(M))
    5 CONTINUE
      S1=DSIN(L*THETAP(N))
      S2=S1*4.0*AR
      Q(MN,KL)=HH1*S2
    2 CONTINUE
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DGMPRD(A,B,R,N,M,L)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(1),B(1),R(1)
C
      IR=0
      IK=-M
      DO 10 K=1,L
      IK=IK+M
      DO 10 J=1,N
      IR=IR+1
      JI=J-N
      IB=IK
      R(IR)=0
      DO 10 I=1,M
      JI=JI+N
      IB=IB+1
   10 R(IR)=R(IR)+A(JI)*B(IB)
      RETURN
      END
```

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C     THIS PROGRAM CALCULATES THE MEAN CAMBER LINE FOR AN IDEAL WING   C
C     GIVEN THE DESIRED CHORDWISE PRESSURE DISTRIBUTION                C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER SP1,RP1,S,SP1D2,T
      INTEGER*4 ITB1(12)/12*0/
      REAL*4 RTB1(28)/28*0.0/
      REAL*8 TITLE1(12)
      EQUIVALENCE (TITLE1,RTB1(5))
      DIMENSION DP(15),Q(15,15),SUMINT(5,5)
      DIMENSION B(15)
      DIMENSION XIP(15)
      DIMENSION WING1(15),WING2(15),WING3(15),WING4(15),WING5(15),XI(15)
      DATA ITB1/1,0,6,8,0,0,0,2,0,0,0,1/
      PI=3.1415927
C
C     INPUT DATA
C
      READ (5,100) SP1,RP1,MESH
      READ (5,200) AR
      READ(5,108) NP
      READ(5,600) TITLE1
      S=SP1-1
      SP1D2=SP1/2
      T=RP1*SP1D2
      READ(5,201)(DP(I),I=1,NP)
C
C     PRINT INPUT DATA
C
      WRITE(6,101)
      WRITE(6,102) SP1,RP1,MESH
      WRITE(6,103) AR
      WRITE(6,104)
      WRITE(6,105)(DP(I),I=1,NP)
C
C
C     COMPUTE THE Q MATRIX
C
      CALL QMAT(NP,AR,Q)
      WRITE(6,300)
      WRITE(6,301)((Q(I,J),J=1,NP),I=1,NP)
C
C     SOLVE FOR THE B VECTOR
C
      DO 3 I=1,NP
```

```
    3 B(I)=DP(I)
      CALL GELG(B,Q,NP,1,1.0E-4,IER2)
      WRITE(6,306)
      WRITE(6,301)(B(I),I=1,NP)
      WRITE(6,501) IER2
C
C     COMPUTE THE CAMBER LINE
C
      CALL CAMBER(SP1,RP1,MESH,AR,SUMINT,B,NP)
      WRITE(6,302)
      WRITE (6,301)((SUMINT(M,N),N=1,SP1D2),M=1,RP1)
C
C     PLOT RESULTS
C
      DO 30 I=1,RP1
      WING1(I)=SUMINT(I,1)
      WING2(I)=SUMINT(I,2)
      WING3(I)=SUMINT(I,3)
      WING4(I)=SUMINT(I,4)
      WING5(I)=SUMINT(I,5)
   30 CONTINUE
      DO 31 I=1,RP1
      PHIP=PI/RP1*(I-0.5)
   31 XI(I)=0.5*(1.0-COS(PHIP))
      DO 32 I=1,NP
      PHIP=PI/NP*(I-0.5)
   32 XIP(I)=0.5*(1.0-COS(PHIP))
      CALL DRAWP(NP,XIP,DP,ITB1,RTB1)
      ITB1(1)=2
      ITB1(2)=1
      CALL DRAWP(RP1,XI,WING1,ITB1,RTB1)
      ITB1(2)=2
      CALL DRAWP(RP1,XI,WING2,ITB1,RTB1)
      ITB1(2)=3
      CALL DRAWP(RP1,XI,WING3,ITB1,RTB1)
      ITB1(2)=4
      CALL DRAWP(RP1,XI,WING4,ITB1,RTB1)
      ITB1(2)=5
      ITB1(1)=3
      CALL DRAWP(RP1,XI,WING5,ITB1,RTB1)
C
  100 FORMAT (3I10)
  101 FORMAT('1',5X,'LIFTING SURFACE SOLUTION FOR A WING')
  102 FORMAT('0',5X,'NO. OF SPANWISE ELEMENTS=',I2,2X,'NO. OF CHORDWISE
     1ELEMENTS=',I2,2X,'INTEGERATION MESH SIZE=',I2)
  103 FORMAT('0',5X,'WING ASPECT RATIO=',F5.2)
  104 FORMAT('0',5X,'THE DESIRED CHORDWISE PRESSURE DISTRIBUTION')
  105 FORMAT( 10(2X,F10.4))
```

```
106 FORMAT('0',5X,'SCALE FACTORS')
107 FORMAT('0',5X,'SCALED PRESSURE DISTRIBUTION')
108 FORMAT(I10)
200 FORMAT (F10.2)
201 FORMAT(8F10.8)
300 FORMAT('0',5X,'THE Q MATRIX')
301 FORMAT('0',5(2X,F10.4))
302 FORMAT('0',5X,'THE CAMBER LINE COORDINATES')
306 FORMAT('0',5X,'THE B VECTOR')
501 FORMAT('0',5X,'ERROR CODE FOR THE B VECTOR ',I4)
600 FORMAT(6A8)
    STOP
    END


    FUNCTION XL(THETA)
    XL=0.0
    RETURN
    END


    FUNCTION C(THETA)
    C=1.0
    RETURN
    END


    FUNCTION DXL(THETA)
    DXL=0.0
    RETURN
    END


    FUNCTION DC(THETA)
    DC=0.0
    DC=1.2732395*COS(THETA)
    RETURN
    END
```

```
      SUBROUTINE QMAT(NP,AR,Q)
C
      DIMENSION Q(15,15)
      PI=3.1415927
      DO 1 M=1,NP
      PHI=PI/NP*(M-0.5)
      DO 2 K=1,NP
      HH1=SIN(K*PHI)
      S1=4.0*AR/C(PI/2.0)
      Q(M,K)=HH1*S1
    2 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE CAMBER(SP1,RP1,MESH,AR,SUMINT,B,NP)
C
      INTEGER RP1,SP1,SP1D2
      DIMENSION SUMINT(5,5)
      DIMENSION B(15)
      PI=3.1415927
      SP1D2=SP1/2
      MCRI=RP1*MESH
      MSSJ=SP1*MESH
      DELPHI=PI/FLOAT(MCRI)
      DTHETA=PI/FLOAT(MSSJ)
C
      DO 1 M=1,RP1
      DO 1 N=1,SP1D2
      PHIP=PI/RP1*(M-0.5)
      THETAP=PI/SP1*(N-0.5)
      XIP=0.5*(1.0-COS(PHIP))
      ETAP=COS(THETAP)
      H1P=XL(THETAP)+C(THETAP)*XIP
C
      SUM=0.0
      SUM5=0.0
      DO 3 I=1,MCRI
      DO 3 J=1,MSSJ
      PHI=PI/FLOAT(MCRI)*(FLOAT(I)-0.5)
      XI=0.5*(1.0-COS(PHI))
      THETA=PI/FLOAT(MSSJ)*(FLOAT(J)-0.5)
      ETA=COS(THETA)
      DETA=ETA-ETAP
      H1=XL(THETA)+C(THETA)*XI
```

```
      H2=DXL(THETA)+(XI*DC(THETA))
      DH1=H1-H1P
      H3=2.0*SIN(THETA)*(DH1-(DETA*H2))
      H4=((2.0/AR*DH1)**2+DETA**2)**1.5
      G1=H3/H4
      G2=-DETA*C(THETA)*SIN(PHI)/H4
      S1=SIN(THETA)
      S2=COS(THETA)
C
      SUM1=0.0
      SUM2=0.0
      DO 2 K=1,NP
      HH1=SIN(K*PHI)*SIN(PHI)/2.0
      SUM1A=HH1*S1*B(K)
      SUM1=SUM1A+SUM1
      IF (K-1) 5,5,6
    5 HH2=0.25*(PHI-SIN(2.0*PHI)/2.0)
      GO TO 14
    6 HH2=0.25*(SIN((K-1)*PHI)/(FLOAT(K)-1.0)-SIN((K+1)*PHI)/(FLOAT(K)+1
     1.0))
   14 CONTINUE
      SUM2A=HH2*S2*B(K)
      SUM2=SUM2A+SUM2
    2 CONTINUE
      SUM1=SUM1*G1
      SUM2=SUM2*G2
      SUM3=SUM1+SUM2
      SUM=SUM3+SUM
      IF (I .LT. MCRI) GO TO 3
      H5=2.0/AR*(XL(THETA)+C(THETA)-H1P)
      H6=(H5**2+DETA**2)**0.5
      G3=-(1.0-H5/H6)/DETA
      SUM4=G3*B(1)*COS(THETA)
      SUM5=SUM4+SUM5
    3 CONTINUE
      SUM=(SUM*DTHETA*DELPHI)/(PI*AR)
      SUM5=0.25*DTHETA*SUM5
      SUMINT(M,N)=SUM+SUM5
    1 CONTINUE
      RETURN
      END
```

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Thesis
P17 Parks 165858
c.1 Lifting surface theory
for wings of arbitrary
planform.